String Topology of Classifying Spaces

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Centre for Symmetry and Deformation

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Joint work with Richard Hepworth

$\mathbf{Z}/2$ coefficients

Part 1: HCFTs

String Topology

String topology studies algebraic structures on $H_*(LX)$.
($LX = \mathrm{map}(S^1, X)$)

Theorem (Chas–Sullivan 1999)

Suppose M^d is a closed manifold. Then $H_{*+d}(LM)$ is a BV algebra.

Theorem (Godin 2007)

Suppose M^d is a closed manifold. Then $H_*(LM)$ is the value on S^1 of a degree d Homological Conformal Field Theory (HCFT).

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Definition of HCFT

Rough Definition

An *HCFT* $\mathcal{F}_*$ *of degree d* is an assignment

$$(\text{1-manifold } X) \mapsto (\text{graded vector space } \mathcal{F}_*(X))$$

$$\begin{pmatrix} \text{cobordism} \\ \Sigma\colon X \to Y \end{pmatrix} \mapsto (H_*(B\mathrm{Diff}(\Sigma)) \otimes \mathcal{F}_*(X) \to \mathcal{F}_{*+d\chi(\Sigma,X)}(Y))$$

compatible with disjoint unions and composition of cobordisms.

In particular, the generator of $H_0(B\mathrm{Diff}(\Sigma))$ induces an operation

$$\mathcal{F}(\Sigma)\colon \mathcal{F}_*(X) \to \mathcal{F}_{*+\mathrm{shift}}(Y).$$

$\mathcal{F}\Big($ $\Big)$ and $\mathcal{F}\Big($ $\Big)$ and make $\mathcal{F}_*(S^1)$ into an algebra and a coalgebra.

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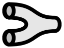
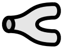
$\mathcal{F}\Big($  $\Big)$ and $\mathcal{F}\Big($  $\Big)$ and make $\mathcal{F}_*(S^1)$ into an algebra and a coalgebra.

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type of cobordisms	open–closed	closed only

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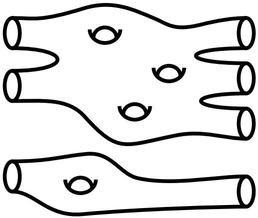

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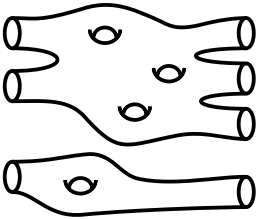

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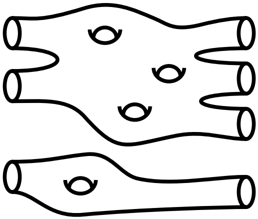

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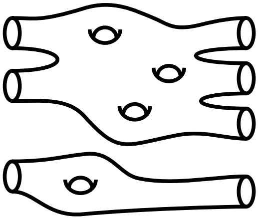

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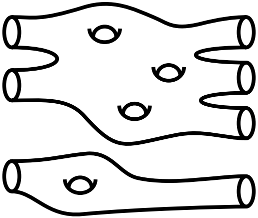

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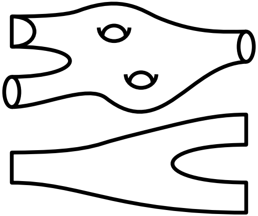

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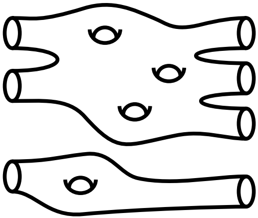

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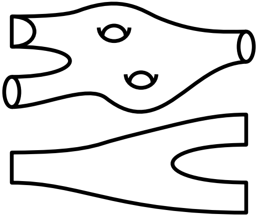

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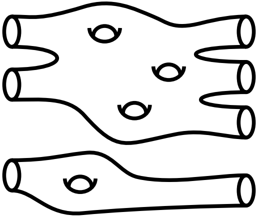

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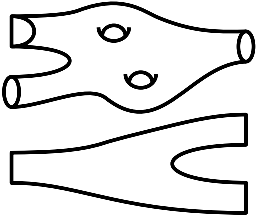

$\mathcal{F}_*(S^1)$ and $\mathcal{F}_*(I)$

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type of cobordisms	open–closed	closed only
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First Theorem

Here is Chataur and Menichi's result again:

Theorem (Chataur and Menichi 2007)

Suppose G is a compact Lie group. Then $H_*(LBG)$ is the value on S^1 an HCFT of degree $-\dim(G)$.

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The HCFT we construct extends Chataur and Menichi's HCFT.

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Q: Is it possible to remove any of the ✗'s in the H–L-column?
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H-Graphs and H-Graph Cobordisms

Work towards the definition a novel kind of field theory – "A Homological H-Graph Field Theory"

Definition

An *h-graph* X is a space homotopy equivalent to a finite graph.

Examples: S^1, $S^1 \vee S^1$, I, connected compact surfaces with non-empty boundary.

Definition

An *h-graph cobordism* $S\colon X \to Y$ is a diagram $X \hookrightarrow S \hookleftarrow Y$ of h-graphs satisfying certain conditions.

Example: An ordinary cobordism $\Sigma\colon X \to Y$ between 1-manifolds with the property that all components of Σ meet X.

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Theorem (Hepworth and L)

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Some Consequences and Benefits

- New cobordisms $\rightsquigarrow$ new operations
  Example:

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- Operations parametrized by homologies of automorphism groups of free groups with boundaries (as well as by homologies of mapping class groups of closed cobordisms)
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  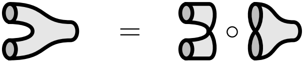

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  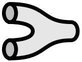 $=$  $\circ$ 

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  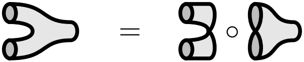

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Other Examples of HHGFTs?

Are there other examples of HHGFTs?

Conjecture

Godin's HCFT extends to an HHGFT.

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