String Topology of Classifying Spaces

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$\mathbf{Z}/2$ coefficients

Part 1: HCFTs

String Topology

String topology studies algebraic structures on $H_*(LX)$.
$(LX = \mathrm{map}(S^1, X))$

Theorem (Chas and Sullivan 1999)

Suppose M is a closed d-manifold. Then $H_{*+d}(LM)$ is a BV algebra.

Theorem (Godin 2007)

Suppose M is a closed d-manifold. Then $H_*(LM)$ is the value on S^1 in a degree d Homological Conformal Field Theory (HCFT).

Definition of HCFT

Rough Definition

An *HCFT* Φ *of degree d* is an assignment

$$\big(\text{1-manifold } X\big) \mapsto \big(\text{graded vector space } \Phi_*(X)\big)$$

$$\begin{pmatrix} \text{cobordism} \\ \Sigma\colon X \to Y \end{pmatrix} \mapsto \big(H_*(B\mathrm{Diff}(\Sigma)) \otimes \Phi_*(X) \to \Phi_{*+d\chi(\Sigma,X)}(Y)\big)$$

compatible with disjoint unions and composition of cobordisms.

Here

$$\mathrm{Diff}(\Sigma) = \left\{ \begin{matrix} \text{orientation-preserving self-diffeos of } \Sigma \\ \text{fixing } X \text{ and } Y \text{ pointwise} \end{matrix} \right\}.$$

Two perspectives on HCFTs

$$\begin{pmatrix}\text{cobordism}\\ \Sigma\colon X\to Y\end{pmatrix} \longmapsto \big(H_*(B\mathrm{Diff}(\Sigma))\otimes\Phi_*(X)\to\Phi_{*+\mathrm{shift}}(Y)\big)$$

1. An HCFT Φ is an algebraic structure on $\Phi_*(S^1)$.
   - A closed 1-manifold X is isomorphic to $\coprod^p S^1$ for some p, and then $\Phi_*(X)\approx\Phi_*(S^1)^{\otimes p}$.
   - Each $z\in H_*(B\mathrm{Diff}(\Sigma))$ for $\Sigma\colon \coprod^p S^1\to\coprod^q S^1$ gives an operation
     $$\Phi_*(S^1)^{\otimes p}\to\Phi_*(S^1)^{\otimes q}.$$
     Eg. the generators of $H_0B\mathrm{Diff}\Big($$\Big)$ and $H_0B\mathrm{Diff}\Big($$\Big)$ make $\Phi_*(S^1)$ into an algebra and a coalgebra.
   - This algebraic structure sheds light on $\Phi_*(S^1)$, for example $H_*(LM)$.

Two perspectives on HCFTs (continued)

$$\binom{\text{cobordism}}{\Sigma\colon X \to Y} \longmapsto \left(H_*(B\mathrm{Diff}(\Sigma)) \otimes \Phi_*(X) \to \Phi_{*+\mathrm{shift}}(Y)\right)$$

2. An HCFT is a 'representation' of $H_*(B\mathrm{Diff}(\Sigma))$'s.
   - $B\mathrm{Diff}(\Sigma)$'s are very interesting spaces: they classify bundles with fibre Σ, and under mild conditions on Σ,
     - $\mathrm{Diff}(\Sigma) \simeq$ mapping class group of Σ
     - $B\mathrm{Diff}(\Sigma) \simeq$ moduli space of Riemann surfaces modelled on Σ.
   - HCFTs give a tool for studying the homology of these spaces.

String Topology of BG

Theorem (Godin 2007)

Suppose M is a closed manifold. Then $H_*(LM)$ is the value on S^1 in an HCFT of degree $\dim(M)$.

Theorem (Chataur and Menichi 2007)

Suppose G is a compact Lie group. Then $H_*(LBG)$ is the value on S^1 in an HCFT of degree $-\dim(G)$.

As stated, the results seem exactly analogous...

Comparison

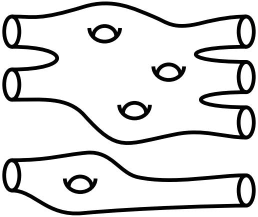
…but in fact they differ significantly in details.

	Godin	C–M
type of cobordisms	open–closed	closed only

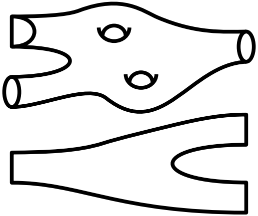
Closed and open–closed cobordisms

Closed: incoming and outgoing boundaries consist of circles (“closed strings”)

Open–closed: also allow intervals (“open strings”) as incoming and outgoing boundaries



$\Phi_*(S^1)$



$\Phi_*(S^1)$ and $\Phi_*(I)$

Comparison (continued)

	Godin	C–M
type of cobordisms	open–closed	closed only
unit for $\Phi_*(S^1)$ 	✓	✗
counit for $\Phi_*(S^1)$ 	✗	✗
unit for $\Phi_*(I)$ 	✓	n/a
counit for $\Phi_*(I)$ 	✓	n/a

First theorem

Here is Chataur and Menichi's result again:

Theorem (Chataur and Menichi 2007)

Suppose G is a compact Lie group. Then $H_*(LBG)$ is the value on S^1 in an HCFT of degree $-\dim(G)$.

Our first theorem is similar:

Theorem (Hepworth and L)

Suppose G is a compact Lie group. Then $H_*(LBG)$ is the value on S^1 in an HCFT of degree $-\dim(G)$.

Comparison

The HCFT we construct extends Chataur and Menichi's HCFT.

	Godin	C–M	H–L
type of cobordisms	open–closed	closed only	open–closed
unit for $\Phi_*(S^1)$ 	✓	✗	✗
counit for $\Phi_*(S^1)$ 	✗	✗	✓
unit for $\Phi_*(I)$ 	✓	n/a	✗
counit for $\Phi_*(I)$ 	✓	n/a	✓

Q: Is it possible to turn any of the ✗'s into ✓'s the H–L-column?
A: No. "Closest possible analogue to Godin's theory."
An extension of C–M to an open–closed theory (without units or counits) has also been constructed by Guldberg. Addition of counits is the harder part – requires new techniques!

Part 2: Beyond HCFTs

Idea: instead of surfaces and diffeomorphisms, work with homotopy graphs and homotopy equivalences.

H-graphs and h-graph cobordisms

Work towards the definition a novel kind of field theory – "A Homological H-Graph Field Theory"

Definition

An *h-graph* X is a space homotopy equivalent to a finite graph.

Examples: S^1, $S^1 \vee S^1$, I, connected compact surfaces with non-empty boundary.

Definition

An *h-graph cobordism* $S\colon X \to Y$ is a diagram $X \hookrightarrow S \hookleftarrow Y$ of h-graphs satisfying certain conditions.

Example: An ordinary cobordism $\Sigma\colon X \to Y$ between 1-manifolds with the property that all components of Σ meet X.

Definition of HHGFT

For an h-graph cobordism $S\colon X \to Y$, denote

$$\mathrm{hAut}(S) = \left\{ \begin{matrix} \text{self-homotopy equivalences of } S \\ \text{fixing } X \text{ and } Y \text{ pointwise} \end{matrix} \right\}.$$

Rough Definition

A *Homological H-Graph Field Theory (HHGFT)* Φ of degree d is an assignment

$$\big(\text{h-graph } X\big) \mapsto \big(\text{graded vector space } \Phi_*(X)\big)$$

$$\begin{pmatrix} \text{h-graph cob} \\ S\colon X \to Y \end{pmatrix} \mapsto \big(H_*(B\mathrm{hAut}(S)) \otimes \Phi_*(X) \to \Phi_{*+d\chi(S,X)}(Y)\big)$$

compatible with disjoint unions and composition of cobordisms.

Second theorem

An HHGFT restricts to an HCFT:

- S^1 and I are h-graphs
- An ordinary cobordism $\Sigma\colon X \to Y$ is an h-graph cobordisms (as long as X meets every component of Σ)
- We have a natural map $\mathrm{Diff}(\Sigma) \to \mathrm{hAut}(\Sigma)$

Theorem (Hepworth and L)

The HCFT from our first theorem extends to an HHGFT Φ^G.

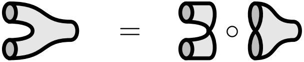
On h-graphs, the theory is given by $\Phi^G_*(X) = H_*\mathrm{map}(X, BG)$.

Some consequences and benefits

- New cobordisms $\rightsquigarrow$ new operations
  Example:

   $: S^1 \to I \quad \rightsquigarrow \quad$ new operation $\Phi_*(S^1) \to \Phi_*(I)$

- New factorizations of existing cobordisms
  Example:

- Operations parametrized by homologies of automorphism groups of free groups (with boundaries)

Automorphism groups of free groups with boundary

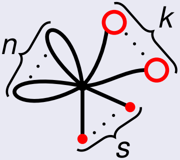
Definition

The *automorphism group of free group on n generators with k boundary circles and s boundary points* is

$$A^s_{n,k} = \pi_0 \mathrm{hAut}(\Gamma^s_{n,k};\, \partial)$$

where

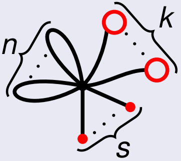
$\Gamma^s_{n,k} =$



and $\partial =$ —.

- $A^1_{n,0} \approx \mathrm{Aut}(F_n)$
- $A^2_{n,0} \approx F_n \rtimes \mathrm{Aut}(F_n) = \mathrm{Hol}(F_n)$
- $A^1_{0,k}$ is a central extension by $\mathbf{Z}^k$ of the pure symmetric automorphism group of F_k.

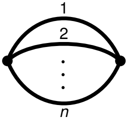
Operations parametrized by $H_*(BA^s_{n,k})$

$A^s_{n,k} = \pi_0 \mathrm{hAut}(\Gamma^s_{n,k}; \partial)$; $\Gamma^s_{n,k} =$ 

; $\partial =$

- If $\partial \neq \emptyset$, we can turn $\Gamma^s_{n,k}$ into an h-graph cobordism by dividing ∂ into incoming and outgoing parts.
- The HHGFT now gives operations parametrized by $H_* B\mathrm{hAut}(\Gamma^s_{n,k}; \partial)$.
- The components of $\mathrm{hAut}(\Gamma^s_{n,k}; \partial)$ are contractible.
- Therefore $\mathrm{hAut}(\Gamma^s_{n,k}; \partial) \simeq A^s_{n,k}$, and we get operations parametrized by $H_*(BA^s_{n,k})$.

Calculations

Let $S_n \colon \mathrm{pt} \to \mathrm{pt}$ be the h-graph cobordism 

.

Σ_n injects into $\mathrm{hAut}(S_n)$ as permutations of the n strings.
Let φ_n^G be the composite

$$\varphi_n^G \colon H_* B\Sigma_n \otimes H_* BG \to H_* B\mathrm{hAut}(S_n) \otimes H_* BG \xrightarrow{\Phi^G(S_n)} H_{*+\mathrm{shift}} BG.$$

Theorem (Hepworth and L)

The map $\varphi_2^{\mathbf{Z}/2} \colon H_* B\Sigma_2 \otimes H_* B(\mathbf{Z}/2) \to H_* B(\mathbf{Z}/2)$
is given by $\quad a \otimes b \longmapsto \begin{cases} a \cdot b & \text{if the degree of } a \text{ is positive} \\ 0 & \text{if the degree of } a \text{ is } 0 \end{cases}$

Recall that $H_* B\Sigma_2$ is a ring: $H_* B\Sigma_2 \approx \Gamma(x)$, $|x| = 1$.
Canonical iso $\mathbf{Z}/2 \approx \Sigma_2$ makes $H_* B(\mathbf{Z}/2)$ into a $H_* B\Sigma_2$-module.

Calculations (continued)

Theorem (Hepworth and L)

The map $\varphi_2^{\mathbf{Z}/2} \colon H_* B\Sigma_2 \otimes H_* B(\mathbf{Z}/2) \to H_* B(\mathbf{Z}/2)$ *is given by*

$$a \otimes b \longmapsto \begin{cases} a \cdot b & \text{if the degree of } a \text{ is positive} \\ 0 & \text{if the degree of } a \text{ is } 0 \end{cases}$$

- Gives an infinite family of non-trivial higher string topology operations – one for each non-zero $a \in H_* B\Sigma_2$, $|a| > 0$.
- For $|a| > 1$, these operations cannot arise from any HCFT operation.

Calculations (continued)

L: Further calculations of $\varphi_n^G\colon H_*B\Sigma_n \otimes H_*BG \to H_{*+\text{shift}}BG$:

- for $G = (\mathbf{Z}/2)^k$, D_{4k+2} and all n
- for $G = \mathbb{T}^k$, $SU(2)$ and small n

Get interesting operations for all these G. For example:

Theorem (L)

The map $\varphi_2^{SU(2)}\colon H_*B\Sigma_2 \otimes H_*BSU(2) \to H_{*+3}BSU(2)$ *is given by* $a_k \otimes b \mapsto a_{k+3} \cdot b$

Here a_k denotes the non-trivial class in $H_kB\Sigma_2 \approx \mathbf{Z}/2$, $k \geq 0$. $H_*BSU(2)$ is made into a $H_*B\Sigma_2$-module using the map

$$\Sigma_2 \times SU(2) \to SU(2), \quad (\sigma, A) \mapsto \begin{cases} A & \text{if } \sigma = \mathrm{id} \\ -A & \text{if } \sigma = (12) \end{cases}$$

Application to homology of $\mathrm{Hol}(F_N)$ and $\mathrm{Aut}(F_N)$

- The calculations give lots of examples of non-trivial string topology operations associated with $S_n \colon \mathrm{pt} \to \mathrm{pt}$.
- More generally, get lots of non-trivial operations for composites $S_{n_1} \circ \cdots \circ S_{n_k} \colon \mathrm{pt} \to \mathrm{pt}$.

Corollary

Get non-trivial classes in

$$H_q B\mathrm{hAut}(S_{n_1} \circ \cdots \circ S_{n_k}) \approx H_q B\mathrm{Hol}(F_N)$$

for various q's. Here $N = \Sigma_{i=1}^{k}(n_i - 1)$.

Corollary

$H_{q-1}(B\mathrm{Aut}(F_N); \tilde{\mathbb{F}}_2^N) \neq 0$ *for these q.*

- For G abelian, the elements in $H_q B\mathrm{Hol}(F_N)$ survive to $H_q B\mathrm{Aff}_N(\mathbf{Z})$, where $\mathrm{Aff}_N(\mathbf{Z}) = \mathrm{Hol}(\mathbf{Z}^N) = \mathbf{Z}^N \rtimes GL_N(\mathbf{Z})$.

Other examples of HHGFTs?

Are there other examples of HHGFTs?

Conjecture

Godin's HCFT in string topology of manifolds extends to an HHGFT.