String topology of finite groups of Lie type

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The main characters

G a compact connected Lie group of dimension d

Two associated objects:

1. finite group of Lie type $G(\mathbb{F}_q)$, $\mathbb{F}_q$ a finite field
2. free loop space $LBG = \mathrm{map}(S^1, BG)$

These may seem disparate mathematical objects ...

The Tezuka conjecture

…but computations show their cohomologies frequently agree.

Let ℓ be a prime $\neq \mathrm{char}(\mathbb{F}_q)$.

Conjecture (Tezuka)

$$H^*(G(\mathbb{F}_q);\mathbb{F}_\ell) \approx H^*(LBG;\mathbb{F}_\ell) \text{ when } q \equiv \begin{cases} 1 \mod \ell & (\ell \text{ odd}) \\ 1 \mod 4 & (\ell = 2) \end{cases}$$

Known to varying degrees of structure when

- $H^*(BG;\mathbb{F}_\ell)$ is polynomial ("the generic case")
- $\ell = 2$, $G = \mathrm{Spin}(n)$

(Tezuka, Kishimoto–Kono, Kameko, Kaji …).

Mysterious! No apparent structural connection between the two sides. This talk: string topology provides such a connection!

The module structure

Write $\mathbb{H}^* := H^{*+d}$. (Recall: $d = \dim(G)$.)

Theorem (Grodal–L)

$H^*(G(\mathbb{F}_q); \mathbb{F}_\ell)$ is a module over $\mathbb{H}^*(LBG; \mathbb{F}_\ell)$ when $\mathbb{H}^*(LBG; \mathbb{F}_\ell)$ is equipped with a string topological multiplication.

No need to assume $q \equiv 1 \bmod \ell$, just $\ell \neq \mathrm{char}(\mathbb{F}_q)$.

A new approach to the Tezuka conjecture: show that the module structure is free of rank 1 when the congruence condition holds.

Theorem (Grodal–L)

The module structure is free of rank 1 when

- *$H^*(BG; \mathbb{F}_\ell)$ is polynomial*
- $\ell = 2$, $G = \mathrm{Spin}(n)$

whenever $q \equiv 1 \bmod \ell$.

The construction I: a space of paths

First step: replace $G(\mathbb{F}_q)$ by a space of paths.

Definition

For X a space and $\sigma\colon X \to X$ a map, the *homotopy fixed point space* of σ is $X^{h\sigma} := \{\alpha\colon I \to X \mid \alpha(1) = \sigma\alpha(0)\}$.

a point in $X^{h\sigma}$

Theorem (Friedlander, Mislin, Quillen)

$BG(\mathbb{F}_q)\hat{}_\ell \simeq (BG\hat{}_\ell)^{h\psi_q}$ for $\psi_q\colon BG\hat{}_\ell \xrightarrow{\simeq} BG\hat{}_\ell$ the q-th unstable Adams operation.

Corollary

$H^*(G(\mathbb{F}_q); \mathbb{F}_\ell) \approx H^*((BG\hat{}_\ell)^{h\psi_q}; \mathbb{F}_\ell)$.

Also, $H^*(LBG; \mathbb{F}_\ell) \approx H^*(L(BG\hat{}_\ell); \mathbb{F}_\ell)$.

The construction II: the product structure

Product on $\mathbb{H}^*(LBG; \mathbb{F}_\ell) = H^{*+d}(L(BG^\wedge_\ell); \mathbb{F}_\ell)$:

Have map $\mathrm{ev}_0 \colon L(BG^\wedge_\ell) \to BG^\wedge_\ell$, $\alpha \mapsto \alpha(0)$.

Diagram

$$\begin{array}{ccccc} L(BG^\wedge_\ell) \times L(BG^\wedge_\ell) & \xleftarrow[*]{\text{split}} & L(BG^\wedge_\ell) \times_{BG^\wedge_\ell} L(BG^\wedge_\ell) & \xrightarrow[!]{\text{concat}} & L(BG^\wedge_\ell) \\ \left(x \xrightarrow{\alpha} x, \; x \xrightarrow{\beta} x \right) & \longleftarrow\!\!\!\mid & \left(x \xrightarrow{\alpha} x \xrightarrow{\beta} x \right) & \longmapsto & \left(x \xrightarrow{\alpha \star \beta} x \right) \end{array}$$

$\rightsquigarrow$ product

$$\circ \colon \mathbb{H}^*(L(BG^\wedge_\ell); \mathbb{F}_\ell) \otimes \mathbb{H}^*(L(BG^\wedge_\ell); \mathbb{F}_\ell) \xrightarrow{\text{concat}_! \circ \text{split}^* \circ \times} \mathbb{H}^*(L(BG^\wedge_\ell); \mathbb{F}_\ell)$$

associative, unital, $H^*(BG^\wedge_\ell; \mathbb{F}_\ell)$-bilinear

The map $\text{concat}_!$ shifts degree by d;
$\mathbb{H}^* = H^{*+d}$ ensures that $\circ$ is degree 0.

The construction III: the module structure

Module structure on $H^*(G(\mathbb{F}_q);\mathbb{F}_\ell) = H^*((BG^{\wedge}_{\ell})^{h\psi_q};\mathbb{F}_\ell)$:

Have map $\mathrm{ev}_0\colon (BG^{\wedge}_{\ell})^{h\psi_q} \to BG^{\wedge}_{\ell}$, $\alpha \mapsto \alpha(0)$.

Diagram

$$\begin{array}{ccccc}
L(BG^{\wedge}_{\ell}) \times (BG^{\wedge}_{\ell})^{h\psi_q} & \xleftarrow[*]{\text{split}} & L(BG^{\wedge}_{\ell}) \times_{BG^{\wedge}_{\ell}} (BG^{\wedge}_{\ell})^{h\psi_q} & \xrightarrow[!]{\text{concat}} & (BG^{\wedge}_{\ell})^{h\psi_q} \\
\left(x \xrightarrow{\alpha} x \,,\; x \xrightarrow{\beta} \psi_q(x) \right) & \longleftarrow\!\!\!\shortmid & \left(x \xrightarrow{\alpha} x \xrightarrow{\beta} \psi_q(x) \right) & \longmapsto & \left(x \xrightarrow{\alpha \star \beta} \psi_q(x) \right)
\end{array}$$

$\rightsquigarrow$ module structure

$$\circ\colon \mathbb{H}^*(L(BG^{\wedge}_{\ell});\mathbb{F}_\ell) \otimes H^*((BG^{\wedge}_{\ell})^{h\psi_q};\mathbb{F}_\ell) \xrightarrow{\text{concat}_! \circ \text{split}^* \circ \times} H^*((BG^{\wedge}_{\ell})^{h\psi_q};\mathbb{F}_\ell)$$

$H^*(BG^{\wedge}_{\ell};\mathbb{F}_\ell)$-bilinear

Remarks

1. The key ingredient: the umkehr maps $\mathrm{concat}_!$. These come from (almost) self-duality of $L(BG\hat{_\ell}) \to BG\hat{_\ell}$ and $(BG\hat{_\ell})^{h\psi_q} \to BG\hat{_\ell}$ as fibrewise $H\mathbb{F}_\ell$-local spectra.
2. Can replace $BG\hat{_\ell}$ with any d-dimensional connected ℓ-compact group BX and ψ_q with any self map $\sigma\colon BX \to BX$:

Theorem (Grodal–L)

$H^*(BX^{h\sigma};\mathbb{F}_\ell)$ is a module over $\mathbb{H}^*(LBX;\mathbb{F}_\ell)$ when $\mathbb{H}^*(LBX;\mathbb{F}_\ell)$ is equipped with a string topological multiplication.

(Work in this generality from now on.)

3. The product on $\mathbb{H}^*(LBX;\mathbb{F}_\ell)$ should agree with the one previously constructed by Chataur and Menichi (with sign corrections by Kuribayashi and Menichi).

Detecting free of rank 1 modules

Write $X := \Omega BX$ (so $X \simeq G^{\wedge}_{\ell}$ if $BX = BG^{\wedge}_{\ell}$).
Have fibre sequence

$$X \xrightarrow{i} BX^{h\sigma} \xrightarrow{\mathrm{ev}_0} BX$$

Theorem (Grodal–L)

$H^*(BX^{h\sigma}; \mathbb{F}_\ell)$ is free of rank 1 as an $\mathbb{H}^*(LBX; \mathbb{F}_\ell)$-module iff $i_*[X] \neq 0 \in H_d(BX^{h\sigma}; \mathbb{F}_\ell)$ for a generator $[X] \in H_d(X; \mathbb{F}_\ell) \approx \mathbb{F}_\ell$.

Translation to case $BX = BG^{\wedge}_{\ell}$, $\sigma = \psi_q$:

$H^*(G(\mathbb{F}_q); \mathbb{F}_\ell)$ is free of rank 1 as an $\mathbb{H}^*(LBG; \mathbb{F}_\ell)$-module iff $i_* \colon H_d(G; \mathbb{F}_\ell) \to H_d(G(\mathbb{F}_q); \mathbb{F}_\ell)$ satisfies $i_*[G] \neq 0$.

Definition

Say $BX^{h\sigma}$ has an $[X]$-*fundamental class* if $i_*[X] \neq 0$.

Spectral sequences

Next: discuss the proof of the detection theorem.

Theorem (Grodal–L)

$H^*(BX^{h\sigma}; \mathbb{F}_\ell)$ is free of rank 1 as an $\mathbb{H}^*(LBX; \mathbb{F}_\ell)$-module iff $BX^{h\sigma}$ has an $[X]$-fundamental class.

Key ingredient: module structure on Serre spectral sequences.

Theorem (Grodal–L)

Write $\mathbb{E}^{*,*} = E^{*,*+d}$.

- (i) *The shifted Serre spectral sequence $\mathbb{E}_r^{*,*}(LBX \to BX)$ is a spectral sequence of algebras and converges to $\mathbb{H}^*(LBX; \mathbb{F}_\ell)$ as an algebra.*
- (ii) *The Serre spectral sequence $E_r^{*,*}(BX^{h\sigma} \to BX)$ is a module spectral sequence over $\mathbb{E}_r^{*,*}(LBX \to BX)$ and converges to $H^*(BX^{h\sigma}; \mathbb{F}_\ell)$ as a module over $\mathbb{H}^*(LBX; \mathbb{F}_\ell)$.*

Proving the detection theorem

Theorem (Grodal–L)

$H^*(BX^{h\sigma}; \mathbb{F}_\ell)$ is free of rank 1 as an $\mathbb{H}^*(LBX; \mathbb{F}_\ell)$-module iff $BX^{h\sigma}$ has an $[X]$-fundamental class.

Sketch of proof of "$\Longleftarrow$".

$\exists[X]$-fundamental class $\implies i_* \neq 0$ on $H_d \implies i^* \neq 0$ on $H^d \implies$ $i^*(x) \neq 0 \in H^d(X; \mathbb{F}_\ell)$ for some $x \in H^d(BX^{h\sigma}; \mathbb{F}_\ell)$. Now

$$z = 1 \otimes i^*(x) \in H^0(BX; \mathbb{F}_\ell) \otimes H^d(X; \mathbb{F}_\ell) = E_2^{0,d}(BX^{h\sigma})$$

is a permanent cycle. Get a map of spectral sequences

$$\mathbb{E}_r^{*,*}(LBX) \xrightarrow{\circ z} E_r^{*,*+d}(BX^{h\sigma}).$$

Check: this is an iso on E_2-pages, hence an iso of SS's. Therefore $\circ x \colon \mathbb{H}^*(LBX; \mathbb{F}_\ell) \to H^{*+d}(BX^{h\sigma}; \mathbb{F}_\ell)$ is an iso, so x gives a basis. □

When is there a fundamental class? I

Theorem (Grodal–L)

$BX^{h\sigma}$ has an $[X]$-fundamental class (and hence the module structure is free of rank 1) when

- *$H^*(BX; \mathbb{F}_\ell)$ is polynomial and σ induces the identity on $H^*(BX; \mathbb{F}_\ell)$*
- *$\ell = 2$, $BX = B\mathrm{Spin}(n)^\wedge_2$ and $\sigma = \psi_q$ for some $q \in \mathbf{Z}_2^\times$.*

Generalizes the theorem from earlier:

Theorem (Grodal–L)

$H^*(G(\mathbb{F}_q); \mathbb{F}_\ell)$ is free of rank 1 over $\mathbb{H}^*(LBG; \mathbb{F}_\ell)$ when

- *$H^*(BG; \mathbb{F}_\ell)$ is polynomial*
- *$\ell = 2$, $G = \mathrm{Spin}(n)$*

whenever $q \equiv 1 \bmod \ell$.

When is there a fundamental class? II

Write $\mathrm{Out}(BX) = \{\sigma\colon BX \xrightarrow{\simeq} BX\}/\simeq$

Theorem (Grodal–L)

For any connected ℓ-compact group BX, the set of $[\sigma] \in \mathrm{Out}(BX)$ for which $BX^{h\sigma}$ has an $[X]$-fundamental class is an uncountable subgroup of

$$\{[\sigma] \in \mathrm{Out}(BX) \mid \sigma \text{ induces the identity on } H^*(BX;\mathbb{F}_\ell)\}.$$

(To show nontriviality of the subgroup, build on Kameko's work)

Optimistic conjecture

$BX^{h\sigma}$ has an $[X]$-fundamental class iff σ induces the identity on $H^*(BX;\mathbb{F}_\ell)$.

How much structure can be preserved?

Suppose $BX^{h\sigma}$ has an $[X]$-fundamental class.
Then $\exists x \in H^d(BX^{h\sigma}; \mathbb{F}_\ell)$ such that the map

$$H^*(LBX; \mathbb{F}_\ell) \xrightarrow[\approx]{\circ x} H^*(BX^{h\sigma}; \mathbb{F}_\ell)$$

is an isomorphism of $H^*(BX; \mathbb{F}_\ell)$-modules

Question

How much more structure can the iso be made to preserve?

Note: the source and target are *not* isomorphic as rings in general! (Example: $\ell = 2$, $BX = B(S^1)^\wedge_2$, $\sigma = \psi_3$.)

Theorem (Grodal–L)

The element x can be chosen so that the iso preserves cup products up to a filtration.

Thank you!

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