Modular characteristic classes for representations over finite fields

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Goals for the talk

- Discuss a new system of modular characteristic classes for group representations over finite fields
- Use it to detect nontrivial classes in $H^*(GL_n(\mathbb{F}_{p^r}); \mathbb{F}_p)$ (for $r = 1$, and mostly for $p = 2$)
- Discuss applications to (co)homology of $\mathrm{Aut}(F_n)$ and $GL_n(\mathbf{Z})$ (if time). Here F_n is the free group on n generators.

Modular characteristic classes?

p a prime, $q = p^r$, $\mathbb{F}$ a field

Definition

A $H^*(-;\,\mathbb{F})$*-valued characteristic class* θ *for representations over* $\mathbb{F}_q$ is an assignment

$$\left(\underbrace{\begin{matrix} G \text{ group,} \\ \rho \text{ representation of } G \text{ over } \mathbb{F}_q \end{matrix}}_{\rho\colon\, G \to GL_n(\mathbb{F}_q)} \right) \longmapsto \theta(\rho) \in H^*(G;\,\mathbb{F}).$$

This assignment must satisfy

1. $\theta(\rho)$ only depends on the isomorphism class of ρ
2. $\theta(f^*\rho) = f^*\theta(\rho)$ for all $f\colon H \to G$. Here $f^*\rho = \rho \circ f$.

Call θ *modular* if $\mathrm{char}(\mathbb{F}) = p$.

Connection to cohomology of $GL_n(\mathbb{F}_q)$

- Characteristic class $\theta \rightsquigarrow$ *universal classes*

$$\theta^{(n)} := \theta(\mathrm{id}_{GL_n(\mathbb{F}_q)}) \in H^*(GL_n(\mathbb{F}_q); \mathbb{F}), \quad n \geq 0$$

- Then $\theta(\rho) = \rho^*(\theta^{(n)})$ when $\dim(\rho) = n$
- Conversely, given elements $\theta^{(n)} \in H^*(GL_n(\mathbb{F}_q); \mathbb{F})$, $n \geq 0$, defining

$$\theta(\rho) := \rho^*(\theta^{(n)}) \qquad \text{if } \dim(\rho) = n$$

  gives a characteristic class
- $\theta^{(n)} \neq 0$ iff $\theta(\rho) \neq 0$ for some n-dimensional rep ρ

Previous results on $H^*(GL_n(\mathbb{F}_q); \mathbb{F})$

Case $\mathrm{char}(\mathbb{F}) \neq p$ is well understood:

- $\mathrm{char}(\mathbb{F}) = 0 \Rightarrow H^*(GL_n(\mathbb{F}_q); \mathbb{F}) = \mathbb{F}$
- Quillen (1972): complete computation of $H^*(GL_n(\mathbb{F}_q); \mathbb{F}_\ell)$, ℓ prime $\neq p$.

Case $\mathrm{char}(\mathbb{F}) = p$ remains poorly understood.

- Quillen: computing $H^*(GL_n(\mathbb{F}_q); \mathbb{F}_p)$ "seems to be a difficult problem once $n \geq 3$."
- First idea: work backwards from $GL(\mathbb{F}_q)$
- Second idea: work backwards from $GL_n(\bar{\mathbb{F}}_q)$
- Neither works! Quillen (1972): $H^*(GL(\mathbb{F}_q); \mathbb{F}_p) = H^*(GL_n(\bar{\mathbb{F}}_q); \mathbb{F}_p) = \mathbb{F}_p$

Results on $H^*(GL_n(\mathbb{F}_{p^r}); \mathbb{F}_p)$

Triviality results: $H^i(GL_n(\mathbb{F}_{p^r}); \mathbb{F}_p) = 0$ when ...

- Friedlander–Parshall (1983): $0 < i < r(2p-3)$
- Quillen (1972) + Maazen (1979): $0 < i < \lfloor n/2 \rfloor$
- Quillen (unpublished): $0 < i < n$ for $p^r \neq 2$

Nontriviality results:

- Computations for $n \leq 4$: Quillen (1972), Aguadé (1980), Tezuka–Yagita (1983),...
- Sprehn (2015): $H^{r(2p-3)}(GL_n(\mathbb{F}_{p^r}); \mathbb{F}_p) \neq 0$ for $2 \leq n \leq p$ (similar earlier results by Barbu (2004), Bendel–Nakano–Pillen (2012))
- Milgram–Priddy (1987): A set of algebraically independent classes of cardinality equal to the Krull dimension of $H^*(GL_n(\mathbb{F}_p); \mathbb{F}_p)$. The classes live in very high degrees (exponential in n).

Summary of results on $H^*(GL_n(\mathbb{F}_{p^r}); \mathbb{F}_p)$

- Our understanding of $H^*(GL_n(\mathbb{F}_{p^r}); \mathbb{F}_p)$ remains very incomplete!
- There is a huge gap between known vanishing range (linear in n) and the degrees of previously known nontrivial classes (exponential in n)
- In our work, we construct explicit nontrivial classes in degrees linear in n

Our characteristic classes

For simplicity, until further notice take $q = p = 2$, $H^* = H^*(-;\mathbb{F}_2)$.

Fact/Convention

$\Sigma_2 \hookrightarrow GL_2(\mathbb{F}_2)$ induces an iso $H^*(GL_2(\mathbb{F}_2)) \xrightarrow{\approx} H^*(\Sigma_2) = \mathbb{F}_2[y]$. We identify $H^*(GL_2(\mathbb{F}_2)) = \mathbb{F}_2[y]$.

Definition

For $k > 0$, let χ_k be the characteristic class defined by the elements $\chi_k^{(n)} = i_! \pi^*(y^k) \in H^k(GL_n(\mathbb{F}_2))$ where

$$GL_n(\mathbb{F}_2) \xleftarrow[\text{incl}]{i} \begin{bmatrix} GL_2 & * \\ & GL_{n-2} \end{bmatrix} \xrightarrow[\text{proj}]{\pi} GL_2(\mathbb{F}_2)$$

(so $\chi_k(\rho) = \rho^*(\chi_k^{(n)})$ when $\dim \rho = n$). We set $\chi_0 = 0$.

($\chi_k(\rho)$ is only defined when $\dim(\rho) \geq 2$.)

Alternative definition

Definition (second version)

Let ρ be a representation over $\mathbb{F}_2$ of a group G with $\dim \rho \geq 2$.

$$\mathrm{pt}/\!\!/ G \xleftarrow{\pi} \mathrm{Emb}(\mathbb{F}_2^2, \rho)/\!\!/ G \times GL_2(\mathbb{F}_2) \xrightarrow{\tau} \mathrm{pt}/\!\!/ GL_2(\mathbb{F}_2)$$

We set $\chi_0(\rho) = 0$ and $\chi_k(\rho) = \pi_! \tau^*(y^k)$ for $k > 0$.

- Emb denotes linear embeddings
- "$/\!\!/$" denotes homotopy orbits: $X /\!\!/ \Gamma = E\Gamma \times_\Gamma X$

The action of $GL_2(\mathbb{F}_2)$ on $\mathrm{Emb}(\mathbb{F}_2^2, \rho)$ is free with orbit space $\mathrm{Gr}_2(\rho)$. So π factors as

$$\mathrm{Emb}(\mathbb{F}_2^2, \rho)/\!\!/ G \times GL_2(\mathbb{F}_2) \xrightarrow{\simeq} \mathrm{Gr}_2(\rho)/\!\!/ G \to \mathrm{pt}/\!\!/ G.$$

Thus $\pi_!$ makes sense. Compatibility of transfers with pullbacks $\Rightarrow \chi_k$ is a characteristic class.

Properties of the characteristic classes

- Vanishing on decomposables:

$$\chi_k(\rho \oplus \eta) = 0$$

  whenever $\dim(\rho), \dim(\eta) > 0$.
- Tensor product formula: Let ρ, η be representations of elementary abelian 2-groups G, H with $\dim(\rho), \dim(\eta) \geq 2$. Then

$$\chi_k(\rho \hat{\otimes} \eta) = \sum_{i+j=k} \binom{k}{i} \chi_i(\rho) \times \chi_j(\eta) \in H^k(G \times H).$$

  ($\rho \hat{\otimes} \eta$ the external tensor product: $(g, h) \cdot v \otimes w = gv \otimes hw$.)
- Nontriviality: $\chi_k(\mathrm{id}_{GL_2\mathbb{F}_2}) = y^k \in H^k(GL_2\mathbb{F}_2)$ for $k > 0$.

The characteristic classes of $\mathbb{F}_2[\Sigma_2]$

Let us compute the characteristic classes of $\mathbb{F}_2[\Sigma_2^n]$.

Lemma

$$\chi_k(\mathbb{F}_2[\Sigma_2]) = \begin{cases} y^k \in H^k(\Sigma_2) & \text{if } k > 0 \\ 0 \in H^0(\Sigma_2) & \text{if } k = 0 \end{cases}$$

Proof.

We have

$$\mathbb{F}_2[\Sigma_2] \approx i^*(\mathrm{id}_{GL_2(\mathbb{F}_2)})$$

where $i\colon \Sigma_2 \to GL_2(\mathbb{F}_2)$ is the inclusion. So

$$\chi_k(\mathbb{F}_2[\Sigma_2]) = i^*\chi_k(\mathrm{id}_{GL_2(\mathbb{F}_2)}) = i^*(y^k) = y^k$$

when $k > 0$. Moreover, $\chi_0 = 0$ by definition. $\square$

The characteristic classes of $\mathbb{F}_2[\Sigma_2^n]$

$H^*(\Sigma_2^n) = \mathbb{F}_2[y_1, \dots, y_n]$ where $y_i = 1 \times \cdots \times \overset{i}{y} \times \cdots \times 1$.

Theorem

$$\chi_k(\mathbb{F}_2[\Sigma_2^n]) = \sum_{\substack{i_1+\cdots+i_n=k \\ i_1,\dots,i_n>0}} \binom{k}{i_1, \dots, i_n} y_1^{i_1} \cdots y_n^{i_n}.$$

Proof.

$$\begin{aligned}
&\chi_k(\mathbb{F}_2[\Sigma_2^n]) \\
&= \chi_k(\mathbb{F}_2[\Sigma_2]^{\hat{\otimes} n}) \\
&= \sum_{i_1+\cdots+i_n=k} \binom{k}{i_1, \dots, i_n} \chi_{i_1}(\mathbb{F}_2[\Sigma_2]) \times \cdots \times \chi_{i_n}(\mathbb{F}_2[\Sigma_2]) \\
&= \sum_{\substack{i_1+\cdots+i_n=k \\ i_1,\dots,i_n>0}} \binom{k}{i_1, \dots, i_n} y_1^{i_1} \cdots y_n^{i_n}
\end{aligned}$$

$\square$

Consequences of the computation

Theorem

$\chi_k(\mathbb{F}_2[\Sigma_2^n]) = \sum_{\substack{i_1+\cdots+i_n=k \\ i_1,\ldots,i_n>0}} \binom{k}{i_1,\ldots,i_n} y_1^{i_1}\cdots y_n^{i_n}$.

Q: When is $\binom{k}{i_1,\ldots,i_n} \neq 0 \mod 2$?
A: Precisely when there is no carry when summing up $i_1,\ldots,i_n$ in binary.

Corollary

$\chi_k(\mathbb{F}_2[\Sigma_2^n]) \neq 0$ *precisely when k has at least n ones in its binary expansion.*

Corollary

Suppose k has at least n ones in its binary expansion. Then $\chi_k^{(2^n)} \neq 0 \in H^k(GL_{2^n}(\mathbb{F}_2))$.

The smallest example is $k = 2^n - 1$. Linearly related to 2^n!

Parabolic induction maps

What about $H^*(GL_d(\mathbb{F}_2))$ when d is not a power of 2?

Definition

For $m \leq n$, the *parabolic induction map* $\Phi_{m,n}$ is

$$\Phi_{m,n} = i_! \circ \pi^* \colon H^*(GL_m(\mathbb{F}_2)) \to H^*(GL_n(\mathbb{F}_2))$$

where

$$GL_n(\mathbb{F}_2) \xleftarrow[\text{incl}]{i} \begin{bmatrix} GL_m & * \\ & GL_{n-m} \end{bmatrix} \xrightarrow[\text{proj}]{\pi} GL_m(\mathbb{F}_2).$$

Example

$\chi_k^{(n)} = \Phi_{2,n}(y^k)$ for $k > 0$.

Lemma

$\Phi_{m,n} \circ \Phi_{\ell,m} = \Phi_{\ell,n}$ *for all* $\ell \leq m \leq n$.

Classes in $H^*(GL_d(\mathbb{F}_2))$

Corollary

$\chi_k^{(N)} = \Phi_{n,N}(\chi_k^{(n)})$ *for all* $n \leq N$.

In particular, $\chi_k^{(N)} \neq 0$ implies that $\chi_k^{(n)} \neq 0$ for all $n \leq N$.
Recall that $\chi_k^{(2^n)} \neq 0$ whenever k has at least n ones in binary.
We get

Theorem

For any $d \geq 2$,

$$\chi_k^{(d)} \neq 0 \in H^k(GL_d(\mathbb{F}_2))$$

whenever the binary expansion of k *has at least* $\log_2(d)$ *ones.*

The smallest example is $k = 2^{\lceil \log_2 d \rceil} - 1 \leq 2d - 3$.
Get nontrivial classes in $H^*(GL_d(\mathbb{F}_2))$ in linear degrees for all d!

Odd primes

We have a $H^*(-;\mathbb{F})$-valued characteristic class χ_α for reps over $\mathbb{F}_{p^r}$ for each $\alpha \in H^*(GL_2(\mathbb{F}_{p^r});\mathbb{F})$. Here char$(\mathbb{F}) = p$. Recall that $H^*(GL_2(\mathbb{F}_2);\mathbb{F}_2) = \mathbb{F}_2[y]$; we have $\chi_k = \chi_{y^k}$.

Fact

For p odd, $H^*(GL_2(\mathbb{F}_p);\ \mathbb{F}_p)$ is the subring of

$$H^*(\mathbb{F}_p;\ \mathbb{F}_p) = \mathbb{F}_p\langle x\rangle \otimes \mathbb{F}_p[y], \ \ |x| = 1, |y| = 2.$$

spanned by monomials $x^a y^b$, $a \in \{0,1\}$, $b \geq 0$, with $p-1 | a+b$.

Theorem

Let p be odd. Then

$$\chi_{x^a y^b}(\mathbb{F}_p[\mathbb{F}_p^n]) = (-1)^{n-1} \sum_{\substack{a_1+\cdots+a_n=a \\ b_1+\cdots+b_n=b \\ p-1|a_i+b_i\neq 0\ \forall i}} \binom{b}{b_1,\ldots,b_n} x^{a_1}y^{b_1}\times\cdots\times x^{a_n}y^{b_n}.$$

Odd primes, continued

Proposition

Let p be odd. Then

$$\chi_{y^m}(\mathbb{F}_p[\mathbb{F}_p^n]) \neq 0 \in H^{2m}(\mathbb{F}_p^n; \mathbb{F}_p)$$

iff the sum of the p-ary digits of m is $k(p-1)$ *for some* $k \geq n$ *and*

$$\chi_{xy^m}(\mathbb{F}_p[\mathbb{F}_p^n]) \neq 0 \in H^{2m+1}(\mathbb{F}_p^n; \mathbb{F}_p)$$

iff the sum of the p-ary digits of m is $k(p-1)-1$ *for some* $k \geq n$*.*

Smallest examples: $m = p^n - 1$ and $m = p^n - p^{n-1} - 1$, respectively.

Corollary

Nontrivial elements $\chi_\alpha^{(d)} \in H^*(GL_d(\mathbb{F}_p); \mathbb{F}_p)$ *in degrees linear in* d*.*

Consequences for $\mathrm{Aut}(F_n)$ and $GL_n(\mathbf{Z})$

The regular representation $\mathbb{F}_p^n \to GL_{p^n}(\mathbb{F}_p)$ factors through many interesting groups. For example:

$$\begin{tikzcd} \mathbb{F}_p^n \arrow[r, "c"] \arrow[rrrr, bend left, "\mathbb{F}_p[\mathbb{F}_p^n]"] & \Sigma_{p^n} \arrow[r, "i"] & \mathrm{Aut}(F_{p^n}) \arrow[r, "\pi_{ab}"] & GL_{p^n}(\mathbf{Z}) \arrow[r, "\rho_{can}"] & GL_{p^n}(\mathbb{F}_p) \end{tikzcd}$$

Corollary

Suppose $\alpha \in H^*(GL_2(\mathbb{F}_p); \mathbb{F}_p)$ *is such that* $\chi_\alpha(\mathbb{F}_p[\mathbb{F}_p^n]) \neq 0$*. Then*

$$\chi_\alpha(\rho_{can}) \neq 0 \in H^*(GL_{p^n}(\mathbf{Z}); \mathbb{F}_p)$$

and

$$\chi_\alpha(\pi_{ab}^* \rho_{can}) \neq 0 \in H^*(\mathrm{Aut}(F_{p^n}); \mathbb{F}_p).$$

Get lots of nontrivial classes. These live in the unstable range where the cohomology of $GL_n(\mathbf{Z})$ and $\mathrm{Aut}(F_n)$ is poorly understood.

Consequences for homology

Computation of $\chi_\alpha(\mathbb{F}_p[F_p^n]) \rightsquigarrow$ indecomposable elements in $H_*(\coprod_n B\mathrm{Aut}(F_n);\, \mathbb{F}_p)$ and $H_*(\coprod_n BGL_n(R);\, \mathbb{F}_p)$, $R = \mathbf{Z}, \mathbb{F}_p$.

Observation

$H_*(\coprod_n B\Sigma_n)$, $H_*(\coprod_n B\mathrm{Aut}(F_n))$ and $H_*(\coprod_n BGL_n(R))$ have ring structures induced by

$$\Sigma_n \times \Sigma_m \xrightarrow{\sqcup} \Sigma_{n+m}$$

$$\mathrm{Aut}(F_n) \times \mathrm{Aut}(F_m) \xrightarrow{*} \mathrm{Aut}(F_{n+m})$$

$$GL_n(R) \times GL_m(R) \xrightarrow{\oplus} GL_{n+m}(R)$$

Moreover, $H_*(\coprod_n B\Sigma_n)$, $H_*(\coprod_n BGL_n(R))$ have an additional product $\circ$ induced by

$$\Sigma_n \times \Sigma_m \xrightarrow{\times} \Sigma_{nm}$$

$$GL_n(R) \times GL_m(R) \xrightarrow{\otimes} GL_{n+m}(R)$$

Consequences for homology, continued

Theorem

Suppose $b_1, \ldots, b_n \in \mathbf{Z}$ are positive multiples of $p-1$ such that there is no carry when $b_1, \ldots, b_n$ are added together in base p. Let $b = b_1 + \cdots + b_n$. Then the following elements are indecomposable in their respective rings:

(i) $i_*(E_{b_1} \circ \cdots \circ E_{b_n}) \in H_{2b}(\mathrm{Aut}(F_{p^n}))$ *in* $H_*(\bigsqcup_k B\mathrm{Aut}(F_k))$

(ii) $E^{\mathbf{Z}}_{b_1} \circ \cdots \circ E^{\mathbf{Z}}_{b_n} \in H_{2b}(GL_{p^n}(\mathbf{Z}))$ *in* $H_*(\bigsqcup_k BGL_k(\mathbf{Z}))$

(iii) $E^{\mathbb{F}_p}_{b_1} \circ \cdots \circ E^{\mathbb{F}_p}_{b_n} \in H_{2b}(GL_{p^n}(\mathbb{F}_p))$ *in* $H_*(\bigsqcup_k BGL_k(\mathbb{F}_p))$.

($H_* = H_*(-;\mathbb{F}_p)$; for $p = 2$, replace $2b$ by b.)

Here $i\colon \Sigma_{p^n} \hookrightarrow \mathrm{Aut}(F_{p^n})$ and $E_k \in H_{2k}(\Sigma_p)$, $E^R_k \in H_{2k}(GL_p(R))$ are certain explicit elements. (For $p = 2$, replace $2k$ by k.)

Consequences for homology, continued

Sketch of proof.

The elements in (i) and (ii) map to the element in (iii) under the ring homomorphisms induced by

$$\mathrm{Aut}(F_{p^n}) \xrightarrow{\pi_{ab}} GL_{p^n}(\mathbf{Z}) \xrightarrow{\rho_{can}} GL_{p^n}(\mathbb{F}_p)$$

so it is enough to prove (iii). We have

$$E_{b_1}^{\mathbb{F}_p} \circ \cdots \circ E_{b_n}^{\mathbb{F}_p} = (\rho_{reg})_*(z_{b_1} \times \cdots \times z_{b_n}).$$

where $z_k \in H_*(\mathbb{F}_p)$ is the dual of $y^k \in H^*(\mathbb{F}_p)$. Thus

$$\langle \chi_{y^b}^{(p^b)}, E_{b_1}^{\mathbb{F}_p} \circ \cdots \circ E_{b_n}^{\mathbb{F}_p} \rangle = \langle \chi_{y^b}(\mathbb{F}_p[\mathbb{F}_p^n]), z_{b_1} \times \cdots \times z_{b_n} \rangle \neq 0.$$

Indecomposability now follows from the vanishing of χ_{y^b} on decomposable representations. □

Thank you!

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