String topology of finite groups of Lie type

Anssi Lahtinen

University of Copenhagen

International Conference on
Manifolds, Groups and Homotopy
Isle of Skye, June 2018

Joint work with Jesper Grodal
Preprint: `http://math.ku.dk/~jg/` or
`https://www.math.uni-hamburg.de/home/lahtinen/`

This project has received funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 800616.

The main characters

G a compact connected Lie group of dimension d

Two associated objects:

1. finite group of Lie type $G(\mathbb{F}_q)$, $\mathbb{F}_q$ a finite field
2. free loop space $LBG = \mathrm{map}(S^1, BG)$

These may seem disparate mathematical objects ...

The Tezuka conjecture

…but computations show their cohomologies frequently agree.

Let ℓ be a prime $\neq \mathrm{char}(\mathbb{F}_q)$.

Conjecture (Tezuka)

$$H^*(G(\mathbb{F}_q); \mathbb{F}_\ell) \approx H^*(LBG; \mathbb{F}_\ell) \textit{ when } q \equiv \begin{cases} 1 \mod \ell & (\ell \textit{ odd}) \\ 1 \mod 4 & (\ell = 2) \end{cases}$$

Known to varying degrees of structure when

- $H^*(BG; \mathbb{F}_\ell)$ is polynomial
- $\ell = 2$, $G = \mathrm{Spin}(n)$

(Tezuka, Kishimoto–Kono, Kameko,…).

Mysterious! No apparent structural connection between the two sides. This talk: string topology provides such a connection!

The module structure

Write $\mathbb{H}^* := H^{*+d}$. (Recall: $d = \dim(G)$.)

Theorem (Grodal–L)

$H^*(G(\mathbb{F}_q); \mathbb{F}_\ell)$ is a module over $\mathbb{H}^*(LBG; \mathbb{F}_\ell)$ when $\mathbb{H}^*(LBG; \mathbb{F}_\ell)$ is equipped with a string topological multiplication.

No need to assume $q \equiv 1 \bmod \ell$, just $\ell \neq \mathrm{char}(\mathbb{F}_q)$.

A new approach to the Tezuka conjecture: show that the module structure is free of rank 1 when the congruence condition holds.

Theorem (Grodal–L)

The module structure is free of rank 1 when

- *$H^*(BG; \mathbb{F}_\ell)$ is polynomial*
- *$\ell = 2$, $G = \mathrm{Spin}(n)$*

whenever $q \equiv 1 \bmod \ell$.

The construction I: a space of paths

First step: replace $G(\mathbb{F}_q)$ by a space of paths.

Definition

For X a space and $\sigma\colon X \to X$ a map, the *homotopy fixed point space* of σ is $X^{h\sigma} := \{\alpha\colon I \to X \mid \alpha(1) = \sigma\alpha(0)\}$.

$x \xrightarrow{\alpha} \sigma(x)$ a point in $X^{h\sigma}$

Theorem (Friedlander, Mislin, Quillen)

$BG(\mathbb{F}_q)\hat{}_\ell \simeq (BG\hat{}_\ell)^{h\psi_q}$ for $\psi_q\colon BG\hat{}_\ell \xrightarrow{\simeq} BG\hat{}_\ell$ the q-th unstable Adams operation.

Corollary

$H^*(G(\mathbb{F}_q); \mathbb{F}_\ell) \approx H^*((BG\hat{}_\ell)^{h\psi_q}; \mathbb{F}_\ell)$.

Also, $H^*(LBG; \mathbb{F}_\ell) \approx H^*(L(BG\hat{}_\ell); \mathbb{F}_\ell)$.

The construction II: the product structure

Product on $\mathbb{H}^*(LBG; \mathbb{F}_\ell) = H^{*+d}(L(BG\hat{_\ell}); \mathbb{F}_\ell)$:

Have map $\mathrm{ev}_0 \colon L(BG\hat{_\ell}) \to BG\hat{_\ell}$, $\alpha \mapsto \alpha(0)$.

Diagram

$$\begin{array}{ccccc} L(BG\hat{_\ell}) \times L(BG\hat{_\ell}) & \xleftarrow[*]{\text{split}} & L(BG\hat{_\ell}) \times_{BG\hat{_\ell}} L(BG\hat{_\ell}) & \xrightarrow[!]{\text{concat}} & L(BG\hat{_\ell}) \\ \left(\overset{x}{\bullet} \xrightarrow{\alpha} \overset{x}{\bullet} , \overset{x}{\bullet} \xrightarrow{\beta} \overset{x}{\bullet} \right) & \longleftarrow\!\!\dashv & \left(\overset{x}{\bullet} \xrightarrow{\alpha} \overset{x}{\bullet} \xrightarrow{\beta} \overset{x}{\bullet} \right) & \longmapsto & \left(\overset{x}{\bullet} \xrightarrow{\alpha \star \beta} \overset{x}{\bullet} \right) \end{array}$$

$\rightsquigarrow$ product

$$\circ \colon \mathbb{H}^*(L(BG\hat{_\ell}); \mathbb{F}_\ell) \otimes \mathbb{H}^*(L(BG\hat{_\ell}); \mathbb{F}_\ell) \xrightarrow{\text{concat}_! \circ \text{split}^* \circ \times} \mathbb{H}^*(L(BG\hat{_\ell}); \mathbb{F}_\ell)$$

associative, unital, $H^*(BG\hat{_\ell}; \mathbb{F}_\ell)$-bilinear

The map $\text{concat}_!$ shifts degree by d;
$\mathbb{H}^* = H^{*+d}$ ensures that $\circ$ is degree 0.

The construction III: the module structure

Module structure on $H^*(G(\mathbb{F}_q);\mathbb{F}_\ell) = H^*((BG^{\wedge}_{\ell})^{h\psi_q};\mathbb{F}_\ell)$:
Have map $\mathrm{ev}_0\colon (BG^{\wedge}_{\ell})^{h\psi_q} \to BG^{\wedge}_{\ell}$, $\alpha \mapsto \alpha(0)$.
Diagram

$$\begin{array}{ccccc}
L(BG^{\wedge}_{\ell}) \times (BG^{\wedge}_{\ell})^{h\psi_q} & \xleftarrow[*]{\text{split}} & L(BG^{\wedge}_{\ell}) \times_{BG^{\wedge}_{\ell}} (BG^{\wedge}_{\ell})^{h\psi_q} & \xrightarrow[!]{\text{concat}} & (BG^{\wedge}_{\ell})^{h\psi_q} \\
\left(x \xrightarrow{\alpha} x \,,\ x \xrightarrow{\beta} \psi_q(x) \right) & \longleftarrow\!\!\shortmid & \left(x \xrightarrow{\alpha} x \xrightarrow{\beta} \psi_q(x) \right) & \longmapsto & \left(x \xrightarrow{\alpha \star \beta} \psi_q(x) \right)
\end{array}$$

$\rightsquigarrow$ module structure

$$\circ\colon \mathbb{H}^*(L(BG^{\wedge}_{\ell});\mathbb{F}_\ell) \otimes H^*((BG^{\wedge}_{\ell})^{h\psi_q};\mathbb{F}_\ell) \xrightarrow{\ \mathrm{concat}_! \circ \mathrm{split}^* \circ \times\ } H^*((BG^{\wedge}_{\ell})^{h\psi_q};\mathbb{F}_\ell)$$

$H^*(BG^{\wedge}_{\ell};\mathbb{F}_\ell)$-bilinear

Remarks

1. The key ingredient: the umkehr maps $\mathrm{concat}_!$. These come from (almost) self-duality of $L(BG\hat{_\ell}) \to BG\hat{_\ell}$ and $(BG\hat{_\ell})^{h\psi_q} \to BG\hat{_\ell}$ as fibrewise $H\mathbb{F}_\ell$-local spectra.
2. Can replace $BG\hat{_\ell}$ with any d-dimensional connected ℓ-compact group BX and ψ_q with any self map $\sigma\colon BX \to BX$:

Theorem (Grodal–L)

$H^*(BX^{h\sigma};\mathbb{F}_\ell)$ is a module over $\mathbb{H}^*(LBX;\mathbb{F}_\ell)$ when $\mathbb{H}^*(LBX;\mathbb{F}_\ell)$ is equipped with a string topological multiplication.

(Work in this generality from now on.)

3. The product on $\mathbb{H}^*(LBX;\mathbb{F}_\ell)$ should agree with the one previously constructed by Chataur and Menichi (with sign corrections by Kuribayashi and Menichi).

Detecting free of rank 1 modules

Write $X := \Omega BX$ (so $X \simeq G^\wedge_\ell$ if $BX = BG^\wedge_\ell$).
Have fibre sequence

$$X \xrightarrow{i} BX^{h\sigma} \xrightarrow{\mathrm{ev}_0} BX$$

Theorem (Grodal–L)

$H^*(BX^{h\sigma}; \mathbb{F}_\ell)$ is free of rank 1 as an $\mathbb{H}^*(LBX; \mathbb{F}_\ell)$-module iff $i_*[X] \neq 0 \in H_d(BX^{h\sigma}; \mathbb{F}_\ell)$ for a generator $[X] \in H_d(X; \mathbb{F}_\ell) \approx \mathbb{F}_\ell$.

Translation to case $BX = BG^\wedge_\ell$, $\sigma = \psi_q$:

$H^*(G(\mathbb{F}_q); \mathbb{F}_\ell)$ is free of rank 1 as an $\mathbb{H}^*(LBG; \mathbb{F}_\ell)$-module iff $i_* : H_d(G; \mathbb{F}_\ell) \to H_d(G(\mathbb{F}_q); \mathbb{F}_\ell)$ satisfies $i_*[G] \neq 0$.

Definition

Say $BX^{h\sigma}$ has an $[X]$-*fundamental class* if $i_*[X] \neq 0$.

Spectral sequences

Next: discuss the proof of the detection theorem.

Theorem (Grodal–L)

$H^*(BX^{h\sigma};\mathbb{F}_\ell)$ is free of rank 1 as an $\mathbb{H}^*(LBX;\mathbb{F}_\ell)$-module iff $BX^{h\sigma}$ has an $[X]$-fundamental class.

Key ingredient: module structure on Serre spectral sequences.

Theorem (Grodal–L)

Write $\mathbb{E}^{*,*} = E^{*,*+d}$.

(i) *The shifted Serre spectral sequence $\mathbb{E}_r^{*,*}(LBX \to BX)$ is a spectral sequence of algebras and converges to $\mathbb{H}^*(LBX;\mathbb{F}_\ell)$ as an algebra.*

(ii) *The Serre spectral sequence $E_r^{*,*}(BX^{h\sigma} \to BX)$ is a module spectral sequence over $\mathbb{E}_r^{*,*}(LBX \to BX)$ and converges to $H^*(BX^{h\sigma};\mathbb{F}_\ell)$ as a module over $\mathbb{H}^*(LBX;\mathbb{F}_\ell)$.*

Proving the detection theorem

Theorem (Grodal–L)

$H^*(BX^{h\sigma};\mathbb{F}_\ell)$ is free of rank 1 as an $\mathbb{H}^*(LBX;\mathbb{F}_\ell)$-module iff $BX^{h\sigma}$ has an $[X]$-fundamental class.

Sketch of proof of "$\Longleftarrow$".

$\exists[X]$-fundamental class $\implies i_* \neq 0$ on $H_d \implies i^* \neq 0$ on $H^d \implies i^*(x) \neq 0 \in H^d(X;\mathbb{F}_\ell)$ for some $x \in H^d(BX^{h\sigma};\mathbb{F}_\ell)$. Now

$$z = 1 \otimes i^*(x) \in H^0(BX;\mathbb{F}_\ell) \otimes H^d(X;\mathbb{F}_\ell) = E_2^{0,d}(BX^{h\sigma})$$

is a permanent cycle. Get a map of spectral sequences

$$\mathbb{E}_r^{*,*}(LBX) \xrightarrow{\circ z} E_r^{*,*+d}(BX^{h\sigma}).$$

Check: this is an iso on E_2-pages, hence an iso of SS's. Therefore $\circ x\colon \mathbb{H}^*(LBX;\mathbb{F}_\ell) \to H^{*+d}(BX^{h\sigma};\mathbb{F}_\ell)$ is an iso, so x gives a basis. $\square$

When is there a fundamental class? I

Theorem (Grodal–L)

$BX^{h\sigma}$ has an $[X]$-fundamental class (and hence the module structure is free of rank 1) when

- *$H^*(BX;\mathbb{F}_\ell)$ is polynomial and σ induces the identity on $H^*(BX;\mathbb{F}_\ell)$*
- *$\ell = 2$, $BX = B\mathrm{Spin}(n)^\wedge_2$ and $\sigma = \psi_q$ for some $q \in \mathbf{Z}_2^\times$.*

Generalizes the theorem from earlier:

Theorem (Grodal–L)

$H^*(G(\mathbb{F}_q);\mathbb{F}_\ell)$ is free of rank 1 over $\mathbb{H}^*(LBG;\mathbb{F}_\ell)$ when

- *$H^*(BG;\mathbb{F}_\ell)$ is polynomial*
- *$\ell = 2$, $G = \mathrm{Spin}(n)$*

whenever $q \equiv 1 \bmod \ell$.

When is there a fundamental class? II

Write $\mathrm{Out}(BX) = \{\sigma\colon BX \xrightarrow{\simeq} BX\}/\simeq$

Theorem (Grodal–L)

For any connected ℓ-compact group BX, the set of $[\sigma] \in \mathrm{Out}(BX)$ for which $BX^{h\sigma}$ has an $[X]$-fundamental class is an uncountable subgroup of

$$\big\{[\sigma] \in \mathrm{Out}(BX) \mid \sigma \text{ induces the identity on } H^*(BX;\mathbb{F}_\ell)\big\}.$$

Optimistic conjecture

$BX^{h\sigma}$ has an $[X]$-fundamental class iff σ induces the identity on $H^*(BX;\mathbb{F}_\ell)$.

How much structure can be preserved?

Suppose $BX^{h\sigma}$ has an $[X]$-fundamental class.
Then $\exists x \in H^d(BX^{h\sigma}; \mathbb{F}_\ell)$ such that the map

$$H^*(LBX; \mathbb{F}_\ell) \xrightarrow[\approx]{\circ x} H^*(BX^{h\sigma}; \mathbb{F}_\ell)$$

is an isomorphism of $H^*(BX; \mathbb{F}_\ell)$-modules

Question

How much more structure can the iso be made to preserve?

Note: the source and target are *not* isomorphic as rings in general! (Example: $\ell = 2$, $BX = B(S^1)\hat{_2}$, $\sigma = \psi_3$.)

Theorem (Grodal–L)

The element x can be chosen so that the iso preserves cup products up to a filtration.

Thank you!

Preprint:

`http://math.ku.dk/~jg/`

`https://www.math.uni-hamburg.de/home/lahtinen/`