INTRODUCTION TO TOPOLOGICAL K-THEORY
EXERCISE SESSION 3

April 19, 2016

Problem 1. For a space X, let $\mathrm{Vect}^1_{\mathbf{R}}(X)$ and $\mathrm{Vect}^1_{\mathbf{C}}(\mathrm{X})$ denote the sets of isomorphism classes of 1-dimensional real and complex vector bundles over X, respectively. Show that tensor product of vector bundles makes $\mathrm{Vect}^1_{\mathbf{R}}(X)$ and $\mathrm{Vect}^1_{\mathbf{C}}(X)$ into abelian groups. If L is a line bundle over X, what are the transition functions of the inverse L^{-1} in terms of those of L?

Problem 2. Let $\xi \to B$ be an n-dimensional real (complex) vector bundle equipped with a Riemannian (resp. Hermitian) metric. Show that every point in B has a neighbourhood U for which there exists a local trivialization of $\xi|U$ compatible with the metric in the sense that for each $b \in U$, it takes the inner product on ξ_b to the standard inner product on $\mathbf{R}^n$ (resp. the standard Hermitian inner product on $\mathbf{C}^n$). Conclude that ξ has a system of transition functions taking values in in the orthogonal group $O(n) \subset GL_n(\mathbf{R})$ (resp. in the unitary group $U(n) \subset GL_n(\mathbf{C})$).

Problem 3. Check that the two constructions of the vector bundle $\mathrm{Hom}(\xi, \zeta)$ discussed in the lectures produce isomorphic results.

Recall that continuous maps $f_0, f_1 \colon X \to Y$ are *homotopic* if there exists a continuous map $h \colon X \times I \to Y$ such that that $h(x, 0) = f_0(x)$ and $h(x, 1) = f_1(x)$ for all $x \in X$. Here I denotes the unit interval $I = [0, 1]$. The following exercises sketch out a quick proof of the following theorem. (With more effort, we will later prove a more general version of the theorem.)

Theorem. *Let X be a compact Hausdorff space, and let ξ be a vector bundle over a space Y. Suppose $f_0, f_1 \colon X \to Y$ are homotopic maps. Then $f_0^*\xi$ and $f_1^*\xi$ are isomorphic.*

First, recall the Tietze extension theorem: if X is a normal space (such as a paracompact space) and $A \subset X$ is a closed subspace, then any continuous function $A \to \mathbf{R}$ can be extended to a continuous function $X \to \mathbf{R}$.

Problem 4. Suppose X is paracompact, and let ξ be a vector bundle over X. Suppose A is a closed subspace of X, and let s be a section of $\xi|A$. Using the Tietze extension theorem, show that s can be extended to a section of ξ.

Problem 5. Let ξ and ζ be vector bundles over a space X. Observe that vector bundle morphisms $\xi \to \zeta$ over X can be identified with sections of $\mathrm{Hom}(\xi, \zeta) \to X$. If X is paracompact and $A \subset X$ is a closed subspace, use the previous problem to show that any isomorphism between $\xi|A$ and $\zeta|A$ can be extended to an isomorphism between $\xi|U$ and $\zeta|U$ for some neighbourhood U of A.

Problem 6. Use the previous problem to show that if X is a compact Hausdorff space and ξ is a vector bundle over $X \times I$, then the vector bundles $\xi|X \times \{0\}$ and $\xi|X \times \{1\}$ are isomorphic. Use this to prove the theorem. *Hint*: compare ξ and $\xi_t \times I$, where $\xi_t \to X$ is the restriction of ξ to $X \times \{t\}$.