INTRODUCTION TO TOPOLOGICAL K-THEORY
EXERCISE SESSION 4

April 26, 2016

Recall from the first exercise session that up to isomorphism, there exist two non-isomorphic 1-dimensional real vector bundles on S^1, namely the trivial line bundle and the Möbius line bundle.

Problem 1. Let X and Y be topological spaces. Prove that homotopy is an equivalence relation on the set of continuous maps from X to Y.

Problem 2. Let X, Y and Z be topological spaces. Prove that composition of maps descends to homotopy classes in the sense that the map

$$\circ\colon [Y, Z] \times [X, Y] \longrightarrow [X, Z], \quad [f] \circ [g] = [f \circ g].$$

is well defined. Use this to prove that homotopy equivalence is an equivalence relation among topological spaces.

Problem 3. Construct an explicit deformation retraction from $\mathbf{R}^n \setminus \{0\}$ to the $(n-1)$-sphere $S^{n-1} = \{x \in \mathbf{R}^n : \|x\| = 1\}$.

Problem 4. Show that S^1 and the unit 2-disk D^2 are not homotopy equivalent.

Problem 5. Which of the following letters are homotopy equivalent to which others?

A C D G I K L N O P Q R T X Y

What about the letter B?

Problem 6. Recall that a retraction from a space X to a subspace A is a continuous map $r\colon X \to A$ such that $r(x) = x$ for all $x \in A$. Prove that there does not exist a retraction from the unit 2-disk D^2 onto S^1.