INTRODUCTION TO TOPOLOGICAL K-THEORY
EXERCISE SESSION 5

May 3, 2016

Recall that the *direct limit topology* on the union $X = \bigcup_{n=1}^{\infty} X_n$ of an ascending sequence of spaces

$$X_1 \subset X_2 \subset X_3 \subset \cdots$$

is the topology where $U \subset X$ is open if and only if $U \cap X_n$ is open in X_n for all n (equivalently, $C \subset X$ is closed if and only if $C \cap X_n$ is closed in X_n for all n).

Problem 1. Show that the subspace topology on each $X_n \subset X$ induced by the direct limit topology on X agrees with the original topology on X_n.

Problem 2. For any space Y, show that a function $f \colon X \to Y$ is continuous with respect to the direct limit topology if and only if the restriction $f| \colon X_n \to Y$ is continuous for all n.

Problem 3. Equip X with the direct limit topology, and let $A \subset X$ be an open or closed subset. Show that the subspace topology on A agrees with the direct limit topology given by the sequence

$$A \cap X_1 \subset A \cap X_2 \subset A \cap X_3 \subset \cdots .$$

Problem 4. Let n and k be non-negative integers. Show that the spaces $\mathrm{Gr}_n(\mathbb{F}^{n+k})$ and $\mathrm{Gr}_k(\mathbb{F}^{n+k})$ are homeomorphic.

Problem 5. Let $i \colon \mathrm{Gr}_k(\mathbb{F}^n) \to \mathrm{Gr}_k(\mathbb{F}^{n+q})$ be the inclusion and let $j \colon \mathrm{Gr}_k(\mathbb{F}^n) \to \mathrm{Gr}_{k+q}(\mathbb{F}^{n+q})$ be the map sending a vector space $V \in \mathrm{Gr}_k(\mathbb{F}^n)$ to the direct sum $V \oplus \mathbb{F}^q \subset \mathbb{F}^n \oplus \mathbb{F}^q = \mathbb{F}^{n+q}$. What are the pullbacks $i^*\gamma^k(\mathbb{F}^{n+q})$ and $j^*\gamma^{k+q}(\mathbb{F}^{n+q})$?

An *H-space* is a space X together with a point $e \in X$ and a map $m \colon X \times X \to X$ having the property that the two maps $m(e, -) \colon X \to X$ and $m(-, e) \colon X \to X$ are homotopic to the identity map of X. The H-space is called *associative* if the composites $m \circ (m \times \mathrm{id}_X)$ and $m \circ (\mathrm{id}_X \times m)$ are homotopic maps $X \times X \times X \to X$.

Problem 6. Observe (or recall from earlier exercise) that tensor product of vector bundles makes $\mathrm{Vect}^1_{\mathbb{F}}(X)$ into a unital associative monoid (in fact, an abelian group). Use the homotopy classification of vector bundles to conclude that $\mathbf{R}\mathrm{P}^\infty$ and $\mathbf{C}\mathrm{P}^\infty$ admit the structure of associative H-spaces. Can you enrich these structures on $\mathbf{R}\mathrm{P}^\infty$ and $\mathbf{C}\mathrm{P}^\infty$ even further?