INTRODUCTION TO TOPOLOGICAL K-THEORY
EXERCISE SESSION 6

May 10, 2016

Problem 1. Show that the canonical line bundle $\gamma^1(\mathbf{R}^n)$ over $\mathbf{R}\mathrm{P}^n$ is nontrivial for all $1 \leq n \leq \infty$. *Hint*: $\mathbf{R}\mathrm{P}^1$ is homeomorphic to S^1.

Problem 2. Recall that a space M is called an n-dimensional topological manifold if it is Hausdorff and every point in M has a neighbourhood homeomorphic to $\mathbf{R}^n$. Prove that the Grassmannians $\mathrm{Gr}_k(\mathbf{R}^n)$ and $\mathrm{Gr}_k(\mathbf{C}^n)$ are topological manifolds of dimensions $k(n-k)$ and $2k(n-k)$, respectively. *Hint*: For $V \in \mathrm{Gr}_k(\mathbf{F}^n)$, consider the neighbourhood $U_V \subset \mathrm{Gr}_k(\mathbf{F}^n)$ consisting of those $W \in \mathrm{Gr}_k(\mathbf{F}^n)$ which surject onto V in the orthogonal projection of $\mathbf{F}^n$ onto V. Construct a homeomorphism between U_V and the space of linear maps from V to $V^\perp$.

Recall from the previous exercise session that for each m and n, the map

$$\perp\colon \mathrm{Gr}_m(\mathbf{F}^{m+n}) \longrightarrow \mathrm{Gr}_n(\mathbf{F}^{m+n}), \quad V \longmapsto V^\perp$$

is a homeomorphism. Write i for the inclusion maps

$$i\colon \mathrm{Gr}_k(\mathbf{F}^n) \hookrightarrow \mathrm{Gr}_k(\mathbf{F}^{n+q})$$

induced by the inclusions $\mathbf{F}^n \hookrightarrow \mathbf{F}^{n+q}$ and j for the maps

$$j\colon \mathrm{Gr}_k(\mathbf{F}^n) \longrightarrow \mathrm{Gr}_{k+q}(\mathbf{F}^{n+q}),$$

sending $V \in \mathrm{Gr}_k(\mathbf{F}^n)$ to the subspace $V \oplus \mathbf{F}^q \subset \mathbf{F}^n \oplus \mathbf{F}^q = \mathbf{F}^{n+q}$.

Problem 3. Define the (real) *bundle dimension* of a space B to be the smallest integer $k \geq 0$ such that composition with the inclusion $\mathrm{Gr}_m(\mathbf{R}^{m+n}) \hookrightarrow \mathrm{Gr}_m(\mathbf{R}^\infty)$ induces a bijection $[B, \mathrm{Gr}_m(\mathbf{R}^{m+n})] \to [B, \mathrm{Gr}_m(\mathbf{R}^\infty)]$ for all $m, n \geq k$. If there is no such k, we say that the bundle dimension of B is infinite. In what follows, let B be a space with finite bundle dimension k.

(a) Show that any numerable vector bundle on B admits a k-dimensional complement.

(b) Show that the inclusion $i\colon \mathrm{Gr}_k(\mathbf{R}^{2k}) \hookrightarrow \mathrm{Gr}_k(\mathbf{R}^{k+n})$ induces a bijection

$$[B, \mathrm{Gr}_k(\mathbf{R}^{2k})] \xrightarrow{\approx} [B, \mathrm{Gr}_k(\mathbf{R}^{k+n})]$$

for all $n \geq k$.

(c) Show that the map $j\colon \mathrm{Gr}_k(\mathbf{R}^{2k}) \to \mathrm{Gr}_n(\mathbf{R}^{k+n})$ induces a bijection

$$[B, \mathrm{Gr}_k(\mathbf{R}^{2k})] \xrightarrow{\approx} [B, \mathrm{Gr}_n(\mathbf{R}^{k+n})]$$

for all $n \geq k$. *Hint*: $\perp$.

(d) Show that any numerable vector bundle ξ of dimension $n \geq k$ over B splits as a direct sum $\xi \approx \eta \oplus \varepsilon^{n-k}$ for some k-dimensional numerable vector bundle η, and that such an η is unique up to isomorphism.

(e) Show that if ξ and ζ are two vector bundles of dimension $\geq k$ over B such that $\xi \oplus \varepsilon^n$ and $\zeta \oplus \varepsilon^n$ are isomorphic for some $n \geq 0$, then ξ and ζ are isomorphic.