INTRODUCTION TO TOPOLOGICAL K-THEORY
EXERCISE SESSION 7

May 24, 2016

Problem 1. Determine $\mathrm{Vect}^n_{\mathbf{R}}(S^1)$ for all $n \geq 1$.

Problem 2. Let

$$D^2_+ = \{(x_1, x_2, x_3) \in S^2 : x_3 \geq 0\}$$

and

$$D^2_- = \{(x_1, x_2, x_3) \in S^2 : x_3 \leq 0\}$$

be the upper and lower hemispheres of the sphere S^2, respectively, and observe that $D^2_+ \cap D^2_- = S^1$. For $x, y \in S^2$ with $y \neq -x$, write $R(x, y)$ for the rotation of $\mathbf{R}^3$ which sends x to y and fixes the subspace perpendicular to x and y. Let e_1, e_2 and e_3 be the standard basis vectors for $\mathbf{R}^3$.

(a) Show that the tangent bundle TS^2 of S^2 can be obtained by the clutching construction applied to the trivial vector bundles $\varepsilon^2_\pm = D^2_\pm \times \mathbf{R}^2$ over $D^2_\pm$ and an isomorphism

$$\varphi\colon \varepsilon^2_+|S^1 \xrightarrow{\approx} \varepsilon^2_-|S^1$$

given by a continuous map $f\colon S^1 \to GL_2(\mathbf{R})$.

(b) Construct explicit trivializations

$$TS^2|D^2_\pm \xrightarrow{\approx} \varepsilon^2_\pm$$

in terms of the functions $x \mapsto R(x, \pm e_3)$.

(c) Use your work from part (b) to find an explicit formula for the clutching map $f\colon S^1 \to GL_2(\mathbf{R})$.

To each map $f\colon S^1 \to S^1$, one can associate an integer $\deg(f)$, the *degree* of f, which, intuitively speaking, counts how many times $f(z)$ wraps around the circle as z goes around the circle once. A prototypical example of a map of degree n is the power map $z \mapsto z^n$. It is a basic result that two maps $f, g\colon S^1 \to S^1$ are homotopic if and only if $\deg(f) = \deg(g)$.

Problem 3. Recall from Problem 3.1 that tensor product of vector bundles makes $\mathrm{Vect}^1_{\mathbf{C}}(X)$ into a group. Determine the group $\mathrm{Vect}^1_{\mathbf{C}}(S^2)$.

Problem 4. Determine $\mathrm{Vect}^2_{\mathbf{R}}(S^2)$.

The following problem gives an alternative construction of the Grothendieck group.

Problem 5. Let M be a commutative monoid with addition $\oplus$. Consider the quotient $\mathrm{Gr}'(M) = F/R$, where F is the free abelian group generated by the set M and R is the normal subgroup of F generated by all elements of the form $x \oplus y - x - y$ for $x, y \in M$. Show that $\mathrm{Gr}'(M)$ is isomorphic to the Grothendieck group $\mathrm{Gr}(M)$.