INTRODUCTION TO TOPOLOGICAL K-THEORY
EXERCISE SESSION 8

May 31, 2016

Problem 1. Let M be a commutative monoid. Show that the map $i\colon M \to \mathrm{Gr}(M)$, $i(x) = [(x, 0)]$ is injective precisely when M has the following *cancellation property*: for every $x, y, z \in M$,

$$x + z = y + z \qquad \text{implies} \qquad x = y.$$

Give an example of a commutative monoid which does not have the cancellation property.

If M and N are commutative semirings, a *homomorphism* of commutative semirings $f\colon M \to N$ is a function of the underlying sets having the following properties: $f(0) = 0$, $f(1) = 1$, $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in M$, where we have written 0 for the additive and 1 for the multiplicative unit elements.

Problem 2. Let $(M, +, \cdot)$ be a commutative semiring. Show that $\mathrm{Gr}(M)$ admits a unique commutative ring structure making the map $i\colon M \to \mathrm{Gr}(M)$ into a homomorphism of commutative semirings. *Hint*: A straightforward way to solve the problem would be to use the formula for the multiplication given in the lecture. Another is to use the universal property of $\mathrm{Gr}(M)$.

Problem 3. Verify that $K(\mathrm{pt}) \approx \mathbf{Z}$ and $KO(\mathrm{pt}) \approx \mathbf{Z}$ as rings.

Problem 4. Compute the ring $K(S^1)$.

Problem 5. Compute the ring $KO(S^1)$.

Problem 6. Suppose X and Y are pointed spaces compact Hausdorff spaces which are pointed homotopy equivalent. Show that $\tilde{K}(X) \approx \tilde{K}(Y)$.