INTRODUCTION TO TOPOLOGICAL K-THEORY
EXERCISE SESSION 11

June 21, 2016

Let X be a compact Hausdorff space. If $\zeta \to X$ is a complex vector bundle and u is an automorphism $\pi^*\zeta \xrightarrow{\approx} \pi^*\zeta$, write $[\zeta, u]$ for the vector bundle $\pi_0^*(\zeta) \cup_u \pi_\infty^*(\zeta)$ over $X \times S^2$. Here

$$\pi\colon X \times S^1 \longrightarrow X, \quad \pi_0\colon X \times D_0^2 \longrightarrow X \quad \text{and} \quad \pi_\infty\colon X \times D_\infty^2 \longrightarrow X$$

are the projections. By the analysis in Lecture 11, the line bundle $H \to S^2$ is isomorphic to $[\varepsilon^1_{\mathrm{pt}}, z^{-1}]$. Recall the relation $H \oplus H \approx H^{\otimes 2} \oplus \varepsilon^1$.

The following problems are meant to help clarify the meaning of the decomposition $\zeta = (\zeta, p)_+ \oplus (\zeta, p)_-$ when p is a linear clutching function.

Problem 1. Suppose $\xi \to X \times S^2$ is isomorphic to $[\zeta, u]$. Show that for all $n \in \mathbf{Z}$, the vector bundle $[\zeta, uz^n]$ is isomorphic to $\xi \otimes \pi_{S^2}^*(H^{-n})$ where $\pi_{S^2}\colon X \times S^2 \to S^2$ is the projection. (Negative tensor powers of a line bundle should be interpreted as positive tensor powers of the dual line bundle.) Conclude that for a monomial clutching function az^n with a an automorphism of ζ, we have $[\zeta, az^n] = \pi_X^*(\zeta) \otimes \pi_{S^2}^*(H^{-n})$. Here $\pi_X\colon X \times S^2 \to X$ is the projection.

Problem 2. Let $p(x, z) = a(x)z + b(x)$ be a linear clutching function for $\zeta \to X$. Show that

$$Q_p'(x) = \frac{1}{2\pi i} \int_{|z|=1} a(x)[a(x)z + b(x)]^{-1} dz$$

defines a projection operator on ζ such that $p(x, z)Q_p(x) = Q_p'(x)p(x, z)$ for all $x \in X$ and $|z| = 1$. (*Hint*: Use an identity proved in the lectures to show that

$$(aw + b)^{-1} a (az + b)^{-1} = (az + b)^{-1} a (aw + b)^{-1}$$

for $z \neq w$.)

Problem 3. Let $(\zeta, p)_+' = \mathrm{Im}\,(Q_p')$ and $(\zeta, p)_-' = \mathrm{Ker}\,(Q_p')$. Then $\zeta = (\zeta, p)_+' \oplus (\zeta, p)_-'$. Conclude from Problem 2 that p restricts to maps

$$p_+(-, z)\colon (\zeta, p)_+ \longrightarrow (\zeta, p)_+' \quad \text{and} \quad p_-(-, z)\colon (\zeta, p)_- \longrightarrow (\zeta, p)_-'.$$

Prove that $p_+(-, z)$ is an isomorphism for all $|z| \geq 1$ (including $z = \infty$) and $p_-(-, z)$ is an isomorphism for all $|z| \leq 1$. (By the assertion that a linear clutching function $cz + d$ is an isomorphism for $z = \infty$ we mean the assertion that c is an isomorphism.) *Hint*: Suppose w is such that $p(x, w)v = 0$ for some $v \in \zeta_x$. Then $|w| \neq 1$. Show that for $|z| = 1$, we have $(a(x)z + b(x))^{-1} a(x) v = (z - w)^{-1} v$, and deduce that

$$Q_p(x)v = \begin{cases} v & \text{if } |w| < 1 \\ 0 & \text{if } |w| > 1 \end{cases}$$

Problem 4. Let $p_+ = a_+ z + b_+$ and $p_- = a_- z + b_-$ where p_+ and p_- are as in Problem 3. Construct a homotopy from p to $a_+ z \oplus b_-$ through linear clutching functions. Deduce that $[\zeta, p] \approx [(\zeta, p)_+, z] \oplus [(\zeta, p)_-, 1]$.

Problem 5. Deduce from Problems 1 and 4 that

$$[\zeta, p] = (\zeta, p)_+ * (2 - H) + (\zeta, p)_- * 1 \in K(X \times S^2).$$