INTRODUCTION TO TOPOLOGICAL K-THEORY
EXERCISE SESSION 12

June 28, 2016

Problem 1. Recall that a *retraction* of a space X onto a subspace A is a continuous map $r\colon X \to A$ such that $r \circ i = \mathrm{id}_A$, where $i\colon A \to X$ is the inclusion. Use K-theory to prove that there does not exist a retraction from the closed n-disk D^n onto S^{n-1}.

Problem 2. Use problem 1 to prove the Brouwer fixed point theorem:

Theorem. *Any continuous map $f\colon D^n \to D^n$ has a fixed point: $f(x) = x$ for some $x \in D^n$.*

Hint: A map f without a fixed point would allow one to construct a retraction from D^n to S^{n-1}. How?

Problem 3. Let $\zeta \to X$ be a complex vector bundle, and let $p\colon \pi^*\zeta \xrightarrow{\approx} \pi^*\zeta$ be a polynomial clutching function of degree $\leq n$. With the notation of problem set 11, show that $[\mathcal{L}^n(\zeta), \mathcal{L}^n(p)] \approx [\zeta, p] \oplus \pi_X^*\zeta^{\oplus n}$. Here $\pi\colon X \times S^1 \to X$ and $\pi_X\colon X \times S^2 \to X$ are the projections.

Problem 4. Let $j\colon \mathbf{Z}[H]/(H-1)^2 \to K(S^2)$ be the ring homomorphism sending the indeterminate H to the class $[H] \in K(S^2)$; notice that j exists by the identity $H \oplus H \approx H^{\otimes 2} \oplus \varepsilon^1$ proved in lecture 11. Without appealing to Bott periodicity, use Problem 3 and Problems 1 and 5 from problem set 11 to show that the composite

$$K(X) \otimes \mathbf{Z}[H]/(H-1)^2 \xrightarrow{\mathrm{id}\otimes j} K(X) \otimes K(S^2) \xrightarrow{*} K(X \times S^2)$$

is a surjection. (Proving this would be a step along the way in an alternative proof of the periodicity theorem. Of course, we know from Bott periodicity that the above composite is in fact an isomorphism. Does your solution indicate how to construct an inverse to the composite map?)