INTRODUCTION TO TOPOLOGICAL K-THEORY
EXERCISE SESSION 13

July 5, 2016

Problem 1. This problem outlines an alternative construction of the positive K-groups and the construction of external products from this point of view. We view $K^*(X)$ as **Z**-graded. Recall from problem 10.5 that external product makes $\tilde{K}^{\leq 0}(S^0) = K^{\leq 0}(\mathrm{pt})$ into a unital graded ring.

(a) Observe that for any X, the external product

$$\tilde{K}^{\leq 0}(S^0) \otimes \tilde{K}^{\leq 0}(X) \xrightarrow{*} \tilde{K}^{\leq 0}(X)$$

makes $\tilde{K}^{\leq 0}(X)$ into a graded $\tilde{K}^{\leq 0}(S^0)$-module.

(b) Show that $\tilde{K}^{\leq 0}(S^0) \approx \mathbf{Z}[b]$ as graded rings where b has degree -2.

(c) Set

$$\tilde{K}^*(X) = \mathbf{Z}[b^{\pm 1}] \otimes_{\mathbf{Z}[b]} \tilde{K}^{\leq 0}(X).$$

Make sense of this object as a **Z**-graded abelian group, and define the n-th K-group $\tilde{K}^n(X)$ to be the degree n part. Show that for $n \leq 0$, this agrees with the previous definition of $\tilde{K}^n(X)$.

(d) Show that the external product

$$\tilde{K}^{\leq 0}(X) \otimes \tilde{K}^{\leq 0}(Y) \xrightarrow{*} \tilde{K}^{\leq 0}(X \wedge Y)$$

factors as a composite

$$\tilde{K}^{\leq 0}(X) \otimes \tilde{K}^{\leq 0}(Y) \longrightarrow \tilde{K}^{\leq 0}(X) \otimes_{\mathbf{Z}[b]} \tilde{K}^{\leq 0}(Y) \xrightarrow{*} \tilde{K}^{\leq 0}(X \wedge Y).$$

(e) Let R be a commutative ring, let M and N be R-modules, and let S be a commutative ring equipped with a homomorphism $R \to S$. Then there is a natural isomorphism

$$S \otimes_R (M \otimes_R N) \approx (S \otimes_R M) \otimes_S (S \otimes_R N).$$

Use this and the previous part to construct an external product

$$\tilde{K}^*(X) \otimes_{\mathbf{Z}[b^{\pm 1}]} \tilde{K}^*(Y) \xrightarrow{*} \tilde{K}^*(X \wedge Y).$$

Problem 2. Suppose X is a path-connected compact Hausdorff space which can be written as the union of n contractible closed subsets. Show that for all $x_1, \ldots, x_n \in \tilde{K}(X)$, the product $x_1 \cdots x_n$ in $\tilde{K}(X)$ is zero. Conclude that in particular the product on $\tilde{K}(\Sigma Y)$ is trivial for any Y.

Problem 3. (The Cayley–Dickson construction of octonions) A $*$-*algebra* over the reals is a real vector space A equipped with a bilinear multiplication $A \times A \to A$ and a linear map $*\colon A \to A$ satisfying $a^{**} = a$ and $(ab)^* = b^* a^*$ for all $a, b \in A$. Show that for a $*$-algebra A, the direct sum $A \oplus A$ has a $*$-algebra structure given by

$$(a, b)(c, d) = (ac - d^* b, da + bc^*) \quad \text{and} \quad (a, b)^* = (a^*, -b).$$

Notice that iterating this construction starting with **R** (with $* = \mathrm{id}$), one recovers the complex numbers **C** and the quaternions **H**. The *octonions* **O** are the result of applying the construction to the quaternions. Show that the octonions satisfy $a^* a = a a^* = |a|^2$ and $|ab| = |a||b|$, where $|\cdot|$ refers to the usual Euclidean norm on $\mathbf{R}^8$. Conclude that the octonions form a division algebra, and that multiplication by an octonion of unit length gives an orthogonal transformation of $\mathbf{R}^8$. Give an example to show that the octonions fail to be associative.