INTRODUCTION TO TOPOLOGICAL K-THEORY
EXERCISE SESSION 14

July 12, 2016

Problem 1. Recall that for a complex vector bundle ξ over X, we defined

$$\lambda_t(\xi) = \sum_{k \geq 0} \lambda^k(\xi) t^k \in K(X)[t].$$

Extend this definition to a map

$$\lambda_t \colon K(X) \longrightarrow K(X)[[t]]$$

satisfying $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$ for all $x, y \in K(X)$. For $x \in K(X)$, define a formal power series $\psi_t(x) \in K(X)[[t]]$ by setting

$$\psi_{-t}(x) = -t \frac{\lambda'_t(x)}{\lambda_t(x)}.$$

Show that the coefficient of t^k in $\psi_t(x)$ is $\psi^k(x)$.

Problem 2. Observe that the product $S^{2n} \times S^{2n}$ can be obtained from the wedge sum $S^{2n} \vee S^{2n}$ by attaching a $4n$-cell along a map $F \colon S^{4n-1} \to S^{2n} \vee S^{2n}$. Let f be the composite of F and the fold map

$$\nabla \colon S^{2n} \vee S^{2n} \longrightarrow S^{2n}$$

which is the identity map on each wedge summand. Show that the Hopf invariant of f is ± 2.

Problem 3. (Five lemma). Suppose

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
\end{array}$$

is a commutative diagram of abelian groups with exact rows.

(a) Assume that α_2 and α_4 are epimorphisms and that α_5 is a monomorphism. Prove that α_3 is an epimorphism.

(b) Assume that α_2 and α_4 are monomorphisms and that α_1 is an epimorphism. Prove that α_3 is a monomorphism.

(c) Conclude that if α_1 is an epimorphism, α_2 and α_4 are isomorphisms, and α_5 is a monomorphism, then α_3 is an isomorphism.

(In typical applications, one knows that α_1, α_2, α_4 and α_5 are isomorphisms, and one uses the lemma to conclude that α_3 is an isomorphism as well.)