A primer on exterior powers

Let $\mathbb{F}$ be a field. Write $\otimes = \otimes_{\mathbb{F}}$

Def 1: Let V be an $\mathbb{F}$-vector space. The k-th *exterior power* of V is the quotient

$$\Lambda^k V = T^k V / \mathfrak{a}_k$$

where $T^k V = V^{\otimes k}$ ($T^0 V = \mathbb{F}$) and $\mathfrak{a}_k \subset V^{\otimes k}$ is the subspace generated by tensors $v_1 \otimes \cdots \otimes v_k$ s.t. $v_i = v_{i+1}$ for some $1 \leq i < k$.

Eg: $\Lambda^0 V = \mathbb{F}$, $\Lambda^1 V = V$.

For a linear map $f : V \to W$, there is an evident induced map

$$f_* = \Lambda^k(f) : \Lambda^k V \longrightarrow \Lambda^k W$$
$$[v_1 \otimes \cdots \otimes v_k] \longmapsto [f(v_1) \otimes \cdots \otimes f(v_k)] .$$

Def 2: Let X, V be $\mathbb{F}$-vector spaces. A k-linear map $f : V^k \longrightarrow X$ is *alternating* if $f(v_1, \ldots, v_k) = 0$ whenever $v_i = v_{i+1}$ for some $1 \leq i < k$.

The following proposition is immediate from the definition of $\Lambda^k V$ and the analogous property for $T^k V$.

Prop 3: For any $\mathbb{F}$-vector space X, composition with the alternating k-linear map

$$\begin{aligned} V^k &\longrightarrow \Lambda^k V \\ (v_1, \dots, v_k) &\longmapsto [v_1 \otimes \dots \otimes v_k] \end{aligned}$$

gives a bijection

$$\{\text{linear maps } \Lambda^k V \to X\} \xrightarrow{\approx} \left\{\begin{matrix}\text{alternating } k\text{-linear} \\ \text{maps } V^k \to X\end{matrix}\right\}.$$

Concatenation of tensors makes the direct sum

$$T^* V = \bigoplus_{k \geq 0} T^k V$$

into a graded ring. It is easy to check that the direct sum

$$\mathfrak{a} = \bigoplus_{k \geq 0} \mathfrak{a}_k \subset T^* V$$

is an ideal, so the direct sum/quotient

$$\begin{aligned} \Lambda^* V := \bigoplus_{k \geq 0} \Lambda^k V &= \bigoplus_{k \geq 0} T^k V / \mathfrak{a}_k \\ &= \bigoplus_{k \geq 0} T^k V \Big/ \bigoplus_{k \geq 0} \mathfrak{a}_k = T^* V / \mathfrak{a} \end{aligned}$$

has an induced graded ring structure. We write $\wedge$ for the multiplication. For $v_1, \dots, v_k \in V$, the element

$$v_1 \wedge \dots \wedge v_k \in \Lambda^k V$$

is then simply the class of $v_1 \otimes \dots \otimes v_k \in T^k V$ in $\Lambda^k V$. From the equation

$$\begin{aligned} 0 = (v+w) \wedge (v+w) &= v \wedge v + v \wedge w + w \wedge v + w \wedge w \\ &= v \wedge w + w \wedge v \end{aligned}$$

we conclude that

$$(*) \qquad v \wedge w = - w \wedge v$$

for all $v, w \in V$. It follows that $\Lambda^* V$ is graded commutative.

Suppose $\{e_i\}_{i\in I}$ is a basis for V, and pick a total order on I. For a finite subset $S\subset I$, we write

$$e_S = e_{i_1}\wedge\cdots\wedge e_{i_k} \in \Lambda^k V$$

where $S=\{i_1,\ldots,i_k\}$ and $i_1<\cdots<i_k$. (We interpret $e_\emptyset = 1 \in \Lambda^0 V$.) Let

$$P_k(I) = \{ S\subset I \mid |S|=k \}.$$

<u>Thm 4</u>: For all $k\geq 0$, the set $\{e_S\}_{S\in P_k(I)}$ is a basis for $\Lambda^k V$.

<u>Pf</u>: Any element of $\Lambda^k V$ can be written as a linear combination of products of the form

$$v_1\wedge\cdots\wedge v_k, \quad v_j\in V.$$

Writing each v_j as a linear combination of the basis vectors e_i, using distributivity, and using the relation $(*)$ to reorder factors, we see that $\Lambda^k V$ is spanned by the vectors e_S, $S\in P_k(I)$.

It remains to show that the vectors

$$e_S, \quad S\in P_k(I)$$

are linearly independent. Suppose

$$(**) \qquad \alpha_{S_1} e_{S_1} + \cdots + \alpha_{S_n} e_{S_n} = 0$$

where $S_1, \dots, S_n$ are distinct elements of $P_k(I)$. Let $J = \bigcup_{i=1}^n S_i$. Then $J = \{j_1, \dots, j_d\}$ for some $j_1, \dots, j_d \in I$, $j_1 < \dots < j_d$. Let $P : V \longrightarrow \mathbb{F}^d$ be the linear map sending e_{j_a} to the a-th standard basis vector of $\mathbb{F}^d$ and e_i to 0 for all $i \in I \setminus J$. Now the composite

$$V^{\times d} \xrightarrow{P^{\times d}} (\mathbb{F}^d)^{\times d} \xrightarrow{\det} \mathbb{F}$$

$$(u_1, \dots, u_d) \longmapsto \det \begin{pmatrix} | & & | \\ u_1 & \cdots & u_d \\ | & & | \end{pmatrix}$$

is an alternating d-linear map sending $(e_{j_1}, \dots, e_{j_d})$ to 1, so we must have $e_J \neq 0$ in $\Lambda^d V$. For each S_i, multiplying $(**)$ by $e_{J \setminus S_i}$ we obtain the equation

$$\pm \alpha_{S_i} e_J = 0,$$

whence $\alpha_{S_i} = 0$. The claim follows. $\square$

Cor 5: Sp. $\dim V = n$. Then

$$\dim \Lambda^k V = \binom{n}{k} \quad \text{for } 0 \leq k \leq n$$

and

$$\Lambda^k V = 0 \quad \text{for } k > n. \quad \square$$

Thm 6: Let V, W be $\mathbb{F}$-vector spaces, and let $i : V \hookrightarrow V \oplus W$, $j : W \hookrightarrow V \oplus W$ be the inclusions. Then the map

$$\bigoplus_{i+j=k} \Lambda^i(V) \otimes \Lambda^j(W) \longrightarrow \Lambda^k(V \oplus W)$$

$$\alpha \otimes \beta \longmapsto i_*(\alpha) \wedge j_*(\beta)$$

is an isomorphism.

<u>Pf</u>: Pick bases $\{e_a\}_{a\in A}$ for V and $\{e_b\}_{b\in B}$ for W. Then $\{i_*(e_a)\}_{a\in A} \amalg \{j_*(e_b)\}_{b\in B}$ is a basis for $V\oplus W$. Pick total orders on A and B, and give $A \amalg B$ the induced total order where $a<b$ for all $a\in A$, $b\in B$. Using Thm 4, it is easy to verify that bases for the domain and the target derived from these bases correspond under the aforementioned map. $\square$