# Introduction to topological K-theory

## A quick word about K-theory in general

K-theory is a vast subject with connections to a multitude of different subfields of mathematics. There are numerous variants of K-theory, eg.

- algebraic K-theory, defined for rings and schemes, relevant to algebraic geometry, algebra, arithmetic, number theory, ... Also geometric topology!
- K-theory of Banach algebras, relevant to analysis, non-commutative geometry, ...
- topological K-theory, defined for topological spaces. This is what this course is about!

History: the first variant was algebraic K-theory, introduced by Grothendieck for work in algebraic geometry ($\sim 1957$). Topological K-theory was introduced by

Atiyah and Hirzebruch ~1959 mimicking Grothendieck's ideas.

Topological K-theory comes in two main variants:

1) complex K-theory $K(X) = KU(X)$
2) real K-theory $KO(X)$.

These are abelian groups constructed out of complex and real vector bundles over $X$, respectively.

## Vector bundles

Idea: A vector bundle over a space $B$ is a collection of vector spaces parametrized by the points of $B$ and banded together by a topology in a coherive way.

Def 1 An n-dimensional real vector bundle over a space $B$ is a map of spaces over a space $B$ is a map of spaces $p : E \to B$ together with a vector space structure (over $\mathbb{R}$) on $p^{-1}(b)$ for each

$b \in B$ such that the following local triviality condition is satisfied: for each $b_0 \in B$, there exists a neighbourhood $U$ of $b_0$ s.t. $\exists$ homeomorphism $h: p^{-1}(U) \xrightarrow{\approx} U \times \mathbb{R}^n$ which, for each $b \in U$, restricts to a linear isomorphism $p^{-1}(b) \xrightarrow{\approx} \{b\} \times \mathbb{R}^n$.

Terminology:

- $B$ is the base space of the vector bundle
- $E$ —"— total space —"—
- $p$ —"— projection —"—
- the vector spaces $p^{-1}(b)$, $b \in B$ are the fibres
- $h$ is a local trivialization of the vector bundle.

Notation

- Often, we denote the vector bundle just by $E$ (leaving the rest of the structure implicit).
- We will often use Greek letters for vector bundles: $\xi, \zeta, \eta, \dots$
- Denote $E_b = p^{-1}(b)$ fibre over $b \in B$.

Complex vector bundles: same definition, just replace $\mathbb{R}$ by $\mathbb{C}$.

Examples

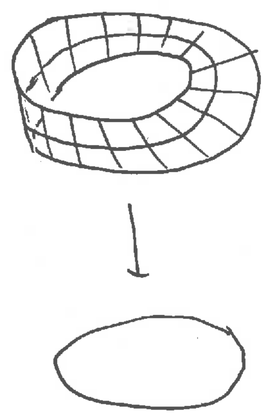
1) the trivial vector bundles

$$\begin{aligned} B \times \mathbb{R}^n &\xrightarrow{p} B \\ (b,v) &\longmapsto b \end{aligned} \quad , \quad \begin{aligned} B \times \mathbb{C}^n &\xrightarrow{p} B \\ (b,v) &\longmapsto b \end{aligned}$$

Notation: $\varepsilon^n_{\mathbb{R}}$, $\varepsilon^n_{\mathbb{C}}$ or just $\varepsilon^n$ (if the choice of $\mathbb{R}$ or $\mathbb{C}$ is clear from context).

2) the Möbius bundle

$$[0,1] \times \mathbb{R} \big/ (0,v) \sim (1,-v) \longrightarrow S^1$$



$$\begin{array}{c} [0,1] \times \mathbb{R} / \sim \\ \big\downarrow \\ S^1 = [0,1]/0 \sim 1 \end{array}$$

3) the tangent bundle of $S^n$:

$$\begin{aligned} TS^n = \{ (x,v) \in S^n \times \mathbb{R}^{n+1} \mid v \perp x \} &\longrightarrow S^n \\ (x,v) &\longmapsto x \end{aligned}$$

(also denoted $T_{S^n}$)

4) the <u>normal bundle</u> of $S^n$ in $\mathbb{R}^{n+1}$:

$$NS^n = \{(x,v) \in S^n \times \mathbb{R}^{n+1} \mid v \perp T_x S^n\} \longrightarrow S^n$$
$$(x,v) \longmapsto x$$

fibre of $TS^n$ over $x$

(also denoted $\nu_{S^n \hookrightarrow \mathbb{R}^{n+1}}$)

5) the <u>tangent bundle</u> $\tau_M = (TM \to M)$ of any smooth manifold $M$

6) the <u>normal bundle</u> $\nu_j$ of any smooth immersion $j : N \looparrowright M$ of smooth manifolds

7) the <u>canonical line bundle</u> over $\mathbb{R}P^n$

$$\mathbb{R}P^n = \{\text{1-dim'l vector subspaces of } \mathbb{R}^{n+1}\}$$
$$= S^n/_{x \sim -x} \qquad \text{(quotient topology)}$$

$$E = \{(\ell, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid v \in \ell\} \longrightarrow \mathbb{R}P^n$$
$$(\ell, v) \longmapsto \ell$$

Note: $E_\ell = \ell$

8) $$E^\perp = \{(\ell, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \mid v \perp \ell\} \longrightarrow \mathbb{R}P^n$$
$$(\ell, v) \longmapsto \ell$$

Note: $E^\perp_\ell = \ell^\perp \subset \mathbb{R}^{n+1}$

<u>Exercise</u>: Find local trivializations for 1-4, 7-8!
(Hint: orthogonal projection)

## Maps between vector bundles

**Def 2** A general map $f = (f_E, f_B)$ of vector bundles from a vector bundle $E_1 \xrightarrow{p_1} B_1$ to a vector bundle $E_2 \xrightarrow{p_2} B_2$ is a commutative diagram

(meaning $p_2 f_E = f_B p_1$)

$$\begin{tikzcd} E_1 \arrow[r, "f_E"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f_B"] & B_2 \end{tikzcd}$$

where $f_B$ and $f_E$ are continuous maps and $f_E$ is linear on fibres: the restriction

$$f_E|_{b_1} : (E_1)_{b_1} \longrightarrow (E_2)_{f_B(b_1)}$$

of $f_E$ is linear $\forall b_1 \in B_1$. The map $f$ (or $f_E$) covers $f_B$ / is over $f_B$.

**Eg** If $\varphi : M \to N$ is a smooth map b/w smooth manifolds, then the differential $d\varphi$ gives a map of vector bundles covering $\varphi$:

$$\begin{tikzcd} TM \arrow[r, "d\varphi"] \arrow[d] & TN \arrow[d] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

Often one restricts to the case $f_B = \mathrm{id}$.

<u>Def 3</u> If $B_1 = B_2 = B$ and $f_B = \mathrm{id}_B$, call $f$ a <u>map of vector bundles over</u> $B$:

$$\begin{tikzcd} E_1 \arrow[rr, "f = f_E"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

Now that we know what maps b/w vector bundles are, we obtain the notion of isomorphism of vector bundles in the usual way.

<u>Def 4</u> A map of vector bundles (over $B$) $f: E_1 \to E_2$ is an <u>isomorphism</u> if $\exists$ map of vector bundles (over $B$) $g: E_2 \to E_1$ s.t. $g \circ f = \mathrm{id}_{E_1}$ and $f \circ g = \mathrm{id}_{E_2}$.

<u>Lemma 5</u> A map $f: E_1 \to E_2$ of vector bundles over $B$ is an isomorphism iff it is an isomorphism on all fibres (i.e. $f|_b : (E_1)_b \to (E_2)_b$ is a linear isomorphism $\forall b \in B$).

Pf: "$\Rightarrow$" is clear.

"$\Leftarrow$": The hypothesis implies that $f$ is a continuous bijection. It remains to show that $f^{-1}: E_2 \to E_1$ is continuous. This can be done locally, so we may restrict to an open $U \subset B$ over which $E_1$ and $E_2$ are trivial. After composing with local trivializations, $f$ amounts to a map of the form

$$\begin{aligned} U \times \mathbb{R}^n &\longrightarrow U \times \mathbb{R}^n \\ (x, v) &\longmapsto (x, A_x v) \end{aligned}$$

where $A_x \in GL_n(\mathbb{R})$ is an invertible matrix depending continuously on $x \in U$. The map $f^{-1}$ then amounts to

$$\begin{aligned} U \times \mathbb{R}^n &\longrightarrow U \times \mathbb{R}^n \\ (x, v) &\longmapsto (x, A_x^{-1} v) \end{aligned}.$$

Since the map $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $A \mapsto A^{-1}$ is continuous, $f^{-1}$ is continuous. $\square$

<u>Def 6</u> A general morphism

$$\begin{tikzcd} E_1 \arrow[r, "f_E"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f_B"] & B_2 \end{tikzcd}$$

is called <u>Cartesian</u> if $f_E$ is an isomorphism on fibres (i.e.

$$f_E|_{b_1} : (E_1)_{b_1} \longrightarrow (E_2)_{f_B(b_1)}$$

is a linear isomorphism $\forall\, b_1 \in B_1$ ).

Notation:

$$\begin{tikzcd} E_1 \arrow[r, "f_E"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f_B"] & B_2 \end{tikzcd}$$

<u>Def 7</u> A <u>pullback</u> of a vector bundle $E \xrightarrow{p} B$ along a continuous map $f: A \to B$ is a vector bundle $f^*E \to A$ together with a Cartesian morphism $f^*E \to E$ covering $f$:

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

(Often the morphism is left implicit.)

<u>Eg</u> If $p: E \to B$ is a vector bundle and $A \subset B$ is a subspace, the <u>restriction</u> of $E$ to $A$, denoted $E|A$, is the vector bundle

$$E|A = p^{-1}(A) \xrightarrow{p|} A.$$

(Exercise: check that this indeed is a vector bundle!) The inclusion $E|A \hookrightarrow E$ exhibits $E|A$ as a pullback of $E$ along the inclusion $A \hookrightarrow B$:

$$\begin{tikzcd} E|A \arrow[r, hook] \arrow[d, "p|"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ A \arrow[r, hook] & B \end{tikzcd}$$

Indeed, the inclusion $E|A \hookrightarrow E$ is the identity map on every fibre.

<u>Next goal:</u> pullbacks of vector bundles along arbitrary maps (not just inclusions of subspaces) always exist & are unique up to unique isomorphism.