Lecture 2

Pullbacks, continued

Recall: A pullback of a v.b. $p: E \to B$ along $f: A \to B$ is a v.b. $f^*E \to A$ together with a Cartesian morphism $f^*E \to E$ covering f:

iso on fibres

$$
\begin{tikzcd}
f^*E \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
$$

(Often the morphism is left implicit.)

Intuition: f^*E is the v.b. over A whose fibre over $a \in A$ "is" the fibre of E over $f(a)$. Here "is" should not be taken literally; it is given content by the map f'.

Eg. If $E \xrightarrow{p} B$ v.b,
$A \subset B$ subspace

then $E|A = p^{-1}(A) \xrightarrow{p|} A$ is a pullback of $E \to B$ along $A \hookrightarrow B$.

$$\begin{tikzcd} E|A \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "p"] \\ A \arrow[r, hook] & B \end{tikzcd}$$

<u>Next goal</u>: Pullbacks along arbitrary maps exist & are unique up to unique isomorphism.

Suppose $E \xrightarrow{p} B$ is a v.b., $f: A \to B$ a map.

<u>Existence of f^*E</u>:

Let

$$A \times_B E = \{(a,v) \in A \times E \mid p(v) = f(a)\} \subset A \times E$$

(<u>fibre product</u> of A and E over B).

Then

$$\begin{tikzcd} A \times_B E \arrow[r, "f'"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad (a,v) \mapsto v, \quad (a,v) \mapsto a$$

commutes. We will show that this diagram gives the desired pullback.

Let $\Gamma_f := \{(a, f(a)) \in A \times B\} \subset A \times B$ (<u>graph</u> of f).

Then p' factors as

vector bundle, restriction of $A \times E \xrightarrow{1 \times p} A \times B$ to Γ_f — homeomorphism

$$\begin{array}{ccccc} A \times_B E & \longrightarrow & \Gamma_f & \xrightarrow{\approx} & A \\ (a, v) & \longmapsto & (a, p(v)) = (a, f(a)) & \longmapsto & a \end{array}$$

So $p' : A \times_B E \longrightarrow A$ is a v.b. Moreover, for any $a_0 \in A$, the map

$$f'|_{a_0} : \underset{\{(a_0, v) \mid v \in E,\ p(v) = f(a_0)\}}{\underset{\|}{(A \times_B E)_{a_0}}} \xrightarrow{(a_0, v) \longmapsto v} \underset{\{v \in E \mid p(v) = f(a_0)\}}{\underset{\|}{E_{f(a_0)}}}$$

is clearly an iso.

<u>Conclusion</u> : can take $f^* E = A \times_B E$.

<u>Uniqueness of $f^* E$</u> :

Suppose

$$\begin{tikzcd} E_1 \arrow[r, "\bar{f}_1"] \arrow[d, "p_1"'] & E \arrow[d, "p"] & E_2 \arrow[l, "\bar{f}_2"'] \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"] & B & B_2 \arrow[l, "f"'] \end{tikzcd}$$

are two pullbacks of $p : E \to B$ along f.

<u>Claim</u> : $\exists$ unique iso $g : E_1 \longrightarrow E_2$ of v.b.'s over A s.t. $\bar{f}_1 = \bar{f}_2 \circ g$.

If g exists, it is necessarily given by

$$(E_1)_a \xrightarrow[\approx]{\bar{f}_1|a} E_{f(a)} \xrightarrow[\approx]{(\bar{f}_2|a)^{-1}} (E_2)_a \qquad (a \in A)$$

on fibres. It remains to show that the map g so defined is continuous. This follows easily by passing to local trivializations. (Do it!) $\square$

The following lemma relates Cartesian morphisms (as defined by us) to how they might be defined in more general contexts.

<u>Lemma 1</u> A general morphism

$$\begin{tikzcd} E_1 \arrow[r, "f_E"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f_B"] & B_2 \end{tikzcd}$$

is Cartesian iff it has the following universal property: Given a v.b. $E \xrightarrow{p} B_1$ and a general morphism $g : E \to E_2$ covering f_B, there exists a unique morphism $h : E \to E_1$ of v.b.'s over B_1 s.t. $g = f_E \circ h$.

$$\begin{tikzcd} E \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "p"'] & & \\ & E_1 \arrow[r, "f_E"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ & B_1 \arrow[r, "f_B"] & B_2 \end{tikzcd}$$

<u>Pf</u>: Exercise. $\square$

Transition functions

Suppose $E \xrightarrow{p} B$ is a v.b. and let $\{U_\alpha\}$ be an open cover of B with local trivializations

$$h_\alpha : p^{-1}(U_\alpha) \xrightarrow{\approx} U_\alpha \times \mathbb{R}^n$$

The composite

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^n \xrightarrow[\approx]{h_\alpha^{-1}|} p^{-1}(U_\alpha \cap U_\beta) \xrightarrow[\approx]{h_\beta|} (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is of the form

$$(x, v) \longmapsto (x, g_{\beta\alpha}(x)v)$$

for some uniquely determined continuous

$$g_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{R})$$

The maps $g_{\beta\alpha}$ are called <u>transition functions</u>.
They satisfy the <u>cocycle condition</u>

$$g_{\gamma\beta}(x)\, g_{\beta\alpha}(x) = g_{\gamma\alpha}(x) \quad \forall x \in U_\alpha \cap U_\beta \cap U_\gamma .$$

We can recover E from the $g_{\beta\alpha}$'s:

Let $E(\{g_{\beta\alpha}\}) := \coprod_\alpha U_\alpha \times \mathbb{R}^n/\sim$

where $\sim$ is defined by setting

$$((x,v) \in U_\alpha \times \mathbb{R}^n) \sim ((x, g_{\beta\alpha}(x)v) \in U_\beta \times \mathbb{R}^n)$$

for all $x \in U_\alpha \cap U_\beta$, $v \in \mathbb{R}^n$.

We have the map

$$\begin{aligned} E(\{g_{\beta\alpha}\}) &\longrightarrow B \\ [(x,v) \in U_\alpha \times \mathbb{R}^n] &\longmapsto x \end{aligned}.$$

<u>Prop 2</u> $E(\{g_{\beta\alpha}\}) \longrightarrow B$ is a v.b. isomorphic to $E \to B$.

<u>Pf</u>: Exercise. $\square$

<u>Conclusion</u>: E can be obtained by gluing together trivial vector bundles in the way described by the transition functions.

We can turn this idea into a method for constructing v.b.'s

<u>Prop 3</u> Given an open cover $\{U_\alpha\}$ of B and continuous functions

$$g_{\beta\alpha}: U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{R})$$

satisfying the cocycle condition, the map $E(\{g_{\beta\alpha}\}) \longrightarrow B$ is an n-dim'l v.b.

<u>Pf</u>: Exercise. $\square$

Constructions on vector bundles

Direct product If $\xi \xrightarrow{p_\xi} B$ and $\zeta \xrightarrow{p_\zeta} C$ are v.b.'s, the *direct product* of ξ and ζ is the v.b.

$$\xi \times \zeta \xrightarrow{p_\xi \times p_\zeta} B \times C.$$

Fibres: $(\xi \times \zeta)_{(b,c)} = \xi_b \times \xi_c$

Get local trivializations by taking products of local trivializations for ξ and ζ.

Direct sum $\xi \xrightarrow{p_\xi} B$, $\zeta \xrightarrow{p_\zeta} B$ v.b.'s.

The *direct sum* or *Whitney sum* of ξ and ζ is

$$\xi \oplus \zeta = \{(v,w) \in \xi \times \zeta \mid p_\xi(v) = p_\zeta(w)\} \longrightarrow B$$
$$(v,w) \longmapsto p_\xi(v) = p_\zeta(w).$$

This is a v.b., since it agrees with

$$(\xi \times \zeta) \times_{B \times B} B = \Delta^*(\xi \times \zeta), \quad \Delta : B \longrightarrow B \times B, \; b \mapsto (b,b) \text{ diagonal}$$

$$\begin{tikzcd} \xi \oplus \zeta \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \xi \times \zeta \arrow[d] \\ B \arrow[r, "\Delta"] & B \times B \end{tikzcd}$$

Fibres: $(\xi \oplus \zeta)_b \approx \xi_b \oplus \zeta_b$.

Generalizing $\oplus$, we have

Metatheorem 4

i) All the usual constructions on vector spaces ($\oplus$, $\otimes$, $\mathrm{Hom}(-,-)$, Λ^k, $(-)^*$, ...) generalize to vector bundles by performing them fibrewise.

ii) Natural isomorphisms b/w these constructions (eg. $V_1 \otimes (V_2 \otimes V_3) \approx (V_1 \otimes V_2) \otimes V_3$, $V^{**} \approx V, \ldots$) generalize to natural isomorphisms b/w the corresponding constructions on v.b.'s

iii) The constructions on v.b.'s commute with taking pullbacks (so eg. $\mathrm{Hom}(f^*\xi, f^*\varsigma) \approx f^*\mathrm{Hom}(\xi,\varsigma)$)

Instead of trying to formalize this and giving a complete proof, let me just discuss an example to illustrate the result.

Suppose $\xi \to B$, $\varsigma \to B$ are v.b.'s of dim. n, m, respectively. We will construct the v.b. $\mathrm{Hom}(\xi,\varsigma) \longrightarrow B$ in two different ways.

First, a bit of notation: if $f: V_2 \to V_1$ and $g: W_1 \to W_2$ are linear maps, write $\mathrm{Hom}(f,g)$ for the linear map

$$\begin{aligned} \mathrm{Hom}(f,g) : \mathrm{Hom}(V_1, W_1) &\longrightarrow \mathrm{Hom}(V_2, W_2) \\ A &\longmapsto g \circ A \circ f \end{aligned}.$$

Construction 1

Choose an open cover $\{U_\alpha\}$ of B and local trivializations of ξ and ζ over each U_α. Let

$$g^{\xi}_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL_n(\mathbb{R})$$

$$g^{\zeta}_{\beta\alpha} : U_\alpha \cap U_\beta \longrightarrow GL_m(\mathbb{R})$$

be the resulting transition functions.

Now the functions

$$\begin{aligned} g_{\beta\alpha} : U_\alpha \cap U_\beta &\longrightarrow \underbrace{GL(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m))}_{\text{linear automorphisms of } \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)} \\ x &\longmapsto \mathrm{Hom}(g^{\xi}_{\beta\alpha}(x)^{-1}, g^{\zeta}_{\beta\alpha}(x)) \end{aligned}$$

are continuous and satisfy the cocycle condition (check!), so they define a v.b. $/B$. This is $\mathrm{Hom}(\xi, \zeta)$.

Rk: The essential properties of $\mathrm{Hom}(-,-)$ (the construction on vector spaces) needed in this construction are

1) Continuity: $\mathrm{Hom}(f,g)$ depends continuously on f and g

2) Functoriality: $\mathrm{Hom}(\mathrm{id}_V, \mathrm{id}_W) = \mathrm{id}_{\mathrm{Hom}(V,W)}$

   and
   $$\mathrm{Hom}(f_2, g_2) \circ \mathrm{Hom}(f_1, g_1) = \mathrm{Hom}(f_1 \circ f_2, g_2 \circ g_1)$$
   for composable linear maps f_1, f_2 and g_2, g_1.
   (Among other things, this implies that $\mathrm{Hom}(f,g)$ is an isomorphism when f and g are.)

Next: the second construction.