# Lecture 3

## Constructions on vector bundles, continued

$\xi \to B$, $\zeta \to B$ v.b.'s of dim. $n, m$, respectively.

Last time: a construction of $\mathrm{Hom}(\xi,\zeta) \to B$ in terms of transition functions.

Let us discuss another way to construct this v.b.

### Construction 2

We want $\mathrm{Hom}(\xi,\zeta)_b \approx \mathrm{Hom}(\xi_b, \zeta_b)$, so let us define $\mathrm{Hom}(\xi,\zeta)$ to be

$$\mathrm{Hom}(\xi,\zeta) = \coprod_{b\in B} \mathrm{Hom}(\xi_b,\zeta_b)$$

as a set, and let $p$ to be the projection

$$p : \mathrm{Hom}(\xi,\zeta) \longrightarrow B$$

which sends $\mathrm{Hom}(\xi_b,\zeta_b)$ to $b\in B$. Then $p^{-1}(b) = \mathrm{Hom}(\xi_b,\zeta_b)$, and what remains is to give $\mathrm{Hom}(\xi,\zeta)$ a topology making $p$ into a vector bundle.

Given an open $U \subset B$ and local triv's

$$h^{\xi} : \xi|U \xrightarrow[\approx]{} U\times\mathbb{R}^n$$

$$h^{\zeta} : \zeta|U \xrightarrow[\approx]{} U\times\mathbb{R}^m,$$

we get a bijection

$$H(h^\xi, h^\zeta) : \mathrm{Hom}(\xi,\zeta)|U \longrightarrow U \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$$
$$A \longmapsto \left(p(A), \mathrm{Hom}\left((h^\xi_{p(A)})^{-1}, h^\zeta_{p(A)}\right)(A)\right)$$

where

$$h^\xi_b : \xi_b \xrightarrow{\approx} \mathbb{R}^n \qquad , \quad b \in U$$
$$h^\zeta_b : \zeta_b \xrightarrow{\approx} \mathbb{R}^m$$

are the maps defined by $h^\xi$ and $h^\zeta$.

Define a topology $T_U$ on $\mathrm{Hom}(\xi,\zeta)|U$ by requiring that $H(h^\xi, h^\zeta)$ is a homeomorphism.

<u>Check</u>:

- $T_U$ is independent of the choice of local trivializations $h^\xi, h^\zeta$ over $U$ (different choices $\rightsquigarrow$ maps $H(h^\xi, h^\zeta)$ related by a homeomorphism of $U \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$)
- If $V \subset U$ is open, $T_V$ agrees with the subspace topology inherited from $T_U$ (can restrict trivializations from $U$ to $V$)

<u>Conclude</u>: The topologies $T_U$ for varying $U$ patch together to give a topology on $\mathrm{Hom}(\xi,\zeta)$ for which $T_U$ agrees with the subspace topology on $\mathrm{Hom}(\xi,\zeta)|U$. Thus $H(h^\xi, h^\zeta)$'s give local trivializations, and $\mathrm{Hom}(\xi,\zeta) \to B$ is a v.b.

Again, the essential features of $\mathrm{Hom}(-,-)$ (the construction on vector spaces) making this work are the continuity and functoriality of $(f,g) \mapsto \mathrm{Hom}(f,g)$.

For the second construction, have

$$\mathrm{Hom}(\xi, \zeta)_b = \mathrm{Hom}(\xi_b, \zeta_b).$$

This makes parts (ii) & (iii) of Metatheorem 4 from Lecture 2 transparent. For example, for (iii), we can define a Cartesian $\bar{f}$

$$\begin{tikzcd} \mathrm{Hom}(f^*\xi, f^*\zeta) \arrow[r, "\bar{f}"] \arrow[d] & \mathrm{Hom}(\xi,\zeta) \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

~~fibres~~ by taking it to be the composite

$$\mathrm{Hom}(f^*\xi, f^*\zeta)_a = \mathrm{Hom}((f^*\xi)_a, (f^*\zeta)_a) \xrightarrow[T]{\approx} \mathrm{Hom}(\xi_{f(a)}, \zeta_{f(a)}) = \mathrm{Hom}(\xi,\zeta)_{f(a)}$$

induced by $(f^*\xi)_a \xrightarrow{\approx} \xi_{f(a)}$
$(f^*\zeta)_a \xrightarrow{\approx} \zeta_{f(a)}$

$(a \in A)$ on fibres.

<u>Exercise</u>: Check that the two constructions of $\mathrm{Hom}(\xi, \zeta)$ agree.

<u>Exercise</u>: Check that when applied to $\oplus$, the analogous constructions recover the Whitney sum of v.b.'s.

## Kernel, image and cokernel

Recall: If $f: V \to W$ is a linear map,

$$\operatorname{Ker} f = \{ v \in V \mid f(v) = 0 \} \subset V$$

$$\operatorname{Im} f = \{ f(v) \mid v \in V \} \subset W$$

$$\operatorname{Coker} f = W / \operatorname{Im} f.$$

We would like to generalize these notions to vector bundles. Suppose $f: \xi \to \zeta$ is a map of vector bundles / $B$. Define

$$\operatorname{Ker} f = \{ v \in \xi \mid f(v) = 0 \} \subset \xi$$

$$\operatorname{Im} f = \{ f(v) \mid v \in \xi \} \subset \zeta$$

$$\operatorname{Coker} f = \zeta / \sim$$ where $w_1 \sim w_2$ if $w_1, w_2 \in \zeta$ belong to the same fibre and $w_1 - w_2 \in \operatorname{Im} f$.

We have evident projections from these spaces to $B$, and the fibres over $b \in B$ are $\operatorname{Ker}(f|_b)$, $\operatorname{Im}(f|_b)$ and $\operatorname{Coker}(f|_b)$, respectively.

Problem: In general, $\nexists$ local trivializations!

Eg:

$$\begin{array}{ccc} (t,v) & \longmapsto & (t, tv) \\ [0,1]\times\mathbb{R} & \xrightarrow{f} & [0,1]\times\mathbb{R} \\ & {}_{pr}\searrow \quad \swarrow_{pr} & \\ & [0,1] & \end{array}$$

The dimensions of $\ker(f|_t)$, $\mathrm{Im}(f|_t)$ and $\mathrm{coker}(f|_t)$ jump at $t=0$, so $\nexists$ local trivializations near $t=0$.

Def A map of v.b.'s $/B$ is of constant rank $k$ if $f|_b : \xi_b \longrightarrow \zeta_b$ is of rank $k$ $\forall b\in B$.

(Recall: the rank of a linear map $f: V\to W$ is $\dim(\mathrm{Im} f) = \dim(V) - \dim(\ker f) = \dim(W) - \dim(\mathrm{coker} f)$ )

Prop Suppose $f: \xi \longrightarrow \zeta$ is of constant rank. Then $\ker f \to B$, $\mathrm{Im} f\to B$ and $\mathrm{coker} f \to B$ are vector bundles.

Pf: It is enough to show local triviality. The problem is local, so we may assume

$$\xi = B\times\mathbb{R}^n, \quad \zeta = B\times \mathbb{R}^m .$$

Then $f:\xi\to\zeta$ has the form

$$\begin{aligned} f: B\times\mathbb{R}^n &\longrightarrow B\times\mathbb{R}^m \\ (b,v) &\longmapsto (b, f_b(v)) \end{aligned}$$

for some continuous $\begin{aligned} B &\longrightarrow \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) . \\ b &\longmapsto f_b \end{aligned}$

Fix $b_0 \in B$ and decompose

$$\mathbb{R}^n = V_1 \oplus V_2 , \qquad V_2 = \operatorname{Ker} f_{b_0}$$

$$\mathbb{R}^m = W_1 \oplus W_2 , \qquad W_1 = \operatorname{Im} f_{b_0} .$$

Let

$$A_b = \left( \begin{array}{c|c} f_b & \begin{matrix} 0 \\ 1 \end{matrix} \\ \hline \begin{matrix} 0 & 1 \end{matrix} & 0 \end{array} \right) : V_1 \oplus V_2 \oplus W_2 \longrightarrow W_1 \oplus W_2 \oplus V_2 .$$

Then the map $B \to \operatorname{Hom}(V_1 \oplus V_2 \oplus W_2, W_1 \oplus W_2 \oplus V_2)$, $b \mapsto A_b$ is continuous and $A_{b_0}$ is an isomorphism, so $A_b$ is an isomorphism for all $b$ in a neighbourhood $U$ of $b_0$.

Let us construct local trivializations over $U$:

<u>Ker $f$</u>: $\begin{cases} A_b(v_1, v_2, 0) = v_2 & \text{for } (v_1, v_2) \in \operatorname{Ker} f_b \\ A_b \text{ invertible} \end{cases}$

$\Rightarrow A_b| : \operatorname{Ker} f_b \longrightarrow V_2$ mono

dimension comparison $\Rightarrow$ —— " —— iso

$$\rightsquigarrow \quad \operatorname{Ker} f|U \xrightarrow{\approx} U \times V_2$$

$$(b, v_1, v_2) \longmapsto (b, A_b(v_1, v_2, 0))$$

$$(b, A_b^{-1}(0,0,v_2)) \longmapsfrom (b, v_2)$$

local triv.

<u>Im f</u>: $\begin{cases} A_b(v_1, 0, 0) = f_b(v_1) & \text{for } v_1 \in V_1 \\ A_b \ \text{invertible} \end{cases}$

$$\Rightarrow A_b| : V_1 \longrightarrow \mathrm{Im}(f_b) \quad \text{mono}$$

$$\overset{\text{dimension comparison}}{\Rightarrow} \quad \text{———} \ \text{''} \ \text{———} \quad \text{iso}$$

$$\rightsquigarrow \begin{array}{ccc} \mathrm{Im}\, f|\mathcal{U} & \xleftarrow{\approx} & \mathcal{U} \times V_1 \\ (b, A_b(v_1,0,0)) & \longleftarrow\!\shortmid & (b, v_1) \\ (b, w_1, w_2) & \longmapsto & (b, A_b^{-1}(w_1, w_2, 0)) \end{array}$$

local triv.

<u>Coker f</u>: $A_b$ restricts to a mono

$$\tilde{A}_b = A_b| : V_1 \oplus W_2 \longrightarrow W_1 \oplus W_2 .$$

Dimension comparison $\Rightarrow$ this is an iso.
Discussion for Im f $\Rightarrow$ under this iso, $V_1 \leftrightarrow \mathrm{Im} f_b$

$$\rightsquigarrow \begin{array}{ccc} \mathcal{U} \times \left( V_1 \oplus W_2 / V_1 \right) & \xrightarrow{\approx} & \mathrm{Coker}\, f|\mathcal{U} \\ (b, (v_1, w_2) + V_1) & \longmapsto & (\tilde{A}_b(v_1, w_2) + \mathrm{Im} f_b) \in (\mathrm{Coker} f)_b \\ (b, \tilde{A}_b^{-1}(w_1, w_2) + V_1) & \longleftarrow\!\shortmid & ((w_1, w_2) + \mathrm{Im} f_b) \in (\mathrm{Coker} f)_b \end{array}$$

local triv. $\square$

Recall: A sequence

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$$

of groups / modules / vector spaces / ... is exact if $\operatorname{Ker} f_2 = \operatorname{Im} f_1$. A sequence

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$$

is exact if it is exact at every $M_i$, $2 \le i \le n-1$.

A short exact sequence is an exact sequence of the form

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

We generalize the language to vector bundles and constant-rank morphisms. In the context of vector bundles / $B$, $0$ should be understood as the $0$-dim'l trivial v.b. / $B$, i.e. $B \xrightarrow{id} B$.

Exercise: If $0 \to \xi \xrightarrow{i} \varsigma \xrightarrow{p} \eta \to 0$ is a short exact sequence of vector bundles / $B$, then $\xi \approx \operatorname{Ker}(p)$ and $\varsigma \approx \operatorname{Coker}(i)$.