Lecture 4

Sections

Def 1: A *section* of a vector bundle $p: \xi \longrightarrow B$ is a continuous map $s: B \to \xi$ s.t. $p \circ s = \mathrm{id}_B$.

Intuition: a section picks out a vector from each fibre in a continuous way.

Notation: $\Gamma(\xi) := \{\text{sections of } \xi\}$.

It makes sense to add sections:

$$(s_1 + s_2)(b) = s_1(b) + s_2(b) \qquad (b \in B)$$

and to multiply a section by a continuous function $f: B \to \mathbb{R}$ ($f: B \to \mathbb{C}$ if ξ is complex):

$$(fs)(b) = f(b)\, s(b) \qquad (b \in B).$$

$\leadsto$ $\Gamma(\xi)$ is a module over the ring

$$C_{\mathbb{R}}(B) = \{ f: B \to \mathbb{R} \mid f \text{ continuous} \}, \ \xi \text{ real}$$

$$C_{\mathbb{C}}(B) = \{ \text{---}''\text{---}\ \mathbb{C} \mid \text{---}''\text{---} \} \ \ \xi \text{ complex}.$$

Rk: These are what correspond to the rings and modules one sees in the def'n of algebraic K-theory.

Def 2: Sections $s_1, \dots, s_k \in \Gamma(\xi)$ are linearly independent if the vectors $s_1(b), \dots, s_k(b) \in \xi_b$ are linearly independent $\forall b \in B$.

Rk: This is a stronger condition than linear independence over the ring $C_{\mathbb{R}}(B)$ (or $C_{\mathbb{C}}(B)$).
Consider, for example, the trivial vector bundle $[-1,1] \times \mathbb{R} \xrightarrow{pr} [-1,1]$ and the section
$s : [-1,1] \longrightarrow [-1,1] \times \mathbb{R}$, $t \mapsto (t, t)$.

Exercise: An n-dimensional v.b. is trivial $\iff$ it admits n linearly independent sections.

Exercise: More generally, a v.b. ξ admits a k-dim'l trivial subbundle $\Leftrightarrow$ ξ has k linearly independent sections.

Partitions of unity, numerability and paracompactness

Def 3: The <u>support</u> of a continuous function $f : X \to \mathbb{R}$ is

$$\mathrm{supp}(f) := \overline{f^{-1}(\mathbb{R} \setminus \{0\})} \subset X .$$

Def 4: A <u>partition of unity</u> (p.o.u.) on a space X is a collection of continuous functions

$$\{\varphi_\alpha : X \to [0,1]\}_{\alpha \in \mathcal{A}}$$

s.t.

(1) Every $x \in X$ has a neighbourhood which intersects $\mathrm{supp}(\varphi_\alpha)$ for only finitely many $\alpha \in \mathcal{A}$ (<u>local finiteness</u>); and

(2) $\sum_{\alpha \in \mathcal{A}} \varphi_\alpha(x) = 1$ for all $x \in X$.

(Note: (1) $\Rightarrow$ the sum is finite $\forall x \in X$)

It is <u>subordinate</u> to an open cover $\{U_\beta\}_{\beta \in \mathcal{B}}$ if $\forall \alpha \in \mathcal{A}\ \exists \beta \in \mathcal{B}$ s.t. $\mathrm{supp}(\varphi_\alpha) \subset U_\beta$.

Def 5: An open cover is <u>numerable</u> if $\exists$ p.o.u. subordinate to it.

Def 6: A Hausdorff space X is <u>paracompact</u> if every open cover of X is numerable.

Rk There is another definition of paracompactness in common use: a Hausdorff space is paracompact if every open cover has a locally finite refinement. The two notions are equivalent. (Def 6 $\Rightarrow$ the other definition by taking the cover $\varphi_\alpha^{-1}(0,1]$, $\alpha \in \mathcal{A}$; for the converse, see eg. [Munkres, Topology, 2nd ed., Thm 41.7].)

Rk about indices: If $\{U_\beta\}_{\beta\in B}$ is a numerable cover of X, $\exists$ p.o.u. $\{\varphi\}_{\beta \in B}$ s.t. $\mathrm{supp}(\varphi_\beta) \subset U_\beta$ $\forall \beta \in B$.

same indexing set!

If $\{\psi_\alpha\}_{\alpha\in\mathcal{A}}$ is a p.o.u. subordinate to $\{U_\beta\}_{\beta\in B}$, pick $f: \mathcal{A} \to B$ s.t. $\mathrm{supp}(\psi_\alpha) \subset U_{f(\alpha)}$ $\forall \alpha \in \mathcal{A}$, and let $\varphi_\beta := \sum_{\alpha\in f^{-1}(\beta)} \psi_\alpha$. Then $\{\varphi_\beta\}_{\beta\in B}$ is as desired. To see this, one uses

Lemma 7: For any $\mathcal{A}' \subset \mathcal{A}$, we have
$$\mathrm{supp}\Big(\sum_{\alpha\in\mathcal{A}'} \varphi_\alpha\Big) = \bigcup_{\alpha\in\mathcal{A}'} \mathrm{supp}(\varphi_\alpha).$$

Pf: Clearly $\mathrm{supp}(\varphi_\alpha) \subset \mathrm{supp}\left(\sum_{\alpha\in\mathcal{A}'} \varphi_\alpha\right)$ $\forall \alpha \in \mathcal{A}'$, so "$\supset$" follows. To show "$\subset$", sp. $x \in \mathrm{supp}\left(\sum_{\alpha\in\mathcal{A}'}\varphi_\alpha\right)$. Pick a neighbourhood V of x meeting $\mathrm{supp}(\varphi_\alpha)$ for only finitely many $\alpha \in \mathcal{A}$. Write
$$\mathcal{A}'' = \{\alpha \in \mathcal{A}' \mid \mathrm{supp}(\varphi_\alpha) \cap V \neq \emptyset\}.$$

By the choice of x, for any neighbourhood W of x, we can find $y \in W \cap V$ s.t.

$$\sum_{\alpha \in \mathcal{U}'} \varphi_\alpha(y) > 0$$

and therefore, since $y \in V$,

$$\sum_{\alpha \in \mathcal{U}''} \varphi_\alpha(y) > 0.$$

Thus
$$\begin{aligned} x \in \operatorname{supp}\left(\sum_{\alpha \in \mathcal{U}''} \varphi_\alpha\right) &= \overline{\left(\sum_{\alpha \in \mathcal{U}''} \varphi_\alpha\right)^{-1}(0,1]} \\ &= \overline{\bigcup_{\alpha \in \mathcal{U}''} \varphi_\alpha^{-1}(0,1]} \\ &= \bigcup_{\alpha \in \mathcal{U}''} \overline{\varphi_\alpha^{-1}(0,1]} \\ &= \bigcup_{\alpha \in \mathcal{U}''} \operatorname{supp}(\varphi_\alpha) \\ &\subset \bigcup_{\alpha \in \mathcal{U}'} \operatorname{supp}(\varphi_\alpha). \end{aligned}$$

The claim follows. □

<u>Def 8</u>: A vector bundle $\xi \to B$ is <u>numerable</u> if $\exists$ numerable cover $\{V_\beta\}_{\beta \in \mathcal{B}}$ of B s.t. $\xi|V_\beta$ is trivial $\forall \beta \in \mathcal{B}$.

<u>Rk</u> Any vector bundle over a paracompact space is numerable.

<u>Rk</u> In vector bundle theory, it is common to work over paracompact base spaces. The main role of such paracompactness assumptions is

to ensure the existence of a p.o.u. one can use to patch together constructions done locally in terms of local trivializations into a global construction. For many purposes, just assuming that the v.b.'s involved are numerable is enough.

<u>Lemma 9</u>: Any two numerable covers of a space have a common numerable refinement.

<u>Pf</u>: Sp. $\{U_\beta\}_{\beta\in\mathcal{B}}$ and $\{V_\gamma\}_{\gamma\in\mathcal{C}}$ are numerable covers of a space X with subordinate p.o.u's $\{\varphi_\beta\}_{\beta\in\mathcal{B}}$ and $\{\psi_\gamma\}_{\gamma\in\mathcal{C}}$, respectively. Then $\{U_\beta\cap V_\gamma\}_{\beta\in\mathcal{B},\gamma\in\mathcal{C}}$ is an open cover refining both $\{U_\beta\}_{\beta\in\mathcal{B}}$ and $\{V_\gamma\}_{\gamma\in\mathcal{C}}$. The products $\varphi_\beta\psi_\gamma$ satisfy

$$\sum_{\substack{\beta\in\mathcal{B}\\ \gamma\in\mathcal{C}}}(\varphi_\beta\psi_\gamma)(x) = \sum_{\substack{\beta\in\mathcal{B}\\ \gamma\in\mathcal{C}}}\varphi_\beta(x)\psi_\gamma(x) = \sum_{\beta\in\mathcal{B}}\varphi_\beta(x)\sum_{\gamma\in\mathcal{C}}\psi_\gamma(x) = 1$$

for all $x\in X$ and

$$\begin{aligned}\mathrm{supp}(\varphi_\beta\psi_\gamma) &= \overline{(\varphi_\beta\psi_\gamma)^{-1}(0,1]}\\ &= \overline{\varphi_\beta^{-1}(0,1]\cap\psi_\gamma^{-1}(0,1]}\\ &\subset \overline{\varphi_\beta^{-1}(0,1]}\cap\overline{\psi_\gamma^{-1}(0,1]}\\ &= \mathrm{supp}(\varphi_\beta)\cap\mathrm{supp}(\psi_\gamma).\end{aligned}$$

In particular, $\mathrm{supp}(\varphi_\beta\psi_\gamma)\subset U_\beta\cap V_\gamma$, and if U (resp. V) is a neighborhood of $x\in X$ meeting

$\mathrm{supp}(\varphi_\beta)$ (resp. $\mathrm{supp}(\psi_\gamma)$) for only finitely many $\beta \in \mathcal{B}$ (resp. $\gamma \in \mathcal{C}$), then $U \cap V$ is a neighbourhood of x meeting $\mathrm{supp}(\varphi_\beta \psi_\gamma)$ for only finitely many $(\beta, \gamma) \in \mathcal{B} \times \mathcal{C}$. Thus $\{\varphi_\beta \psi_\gamma\}_{\beta,\gamma}$ is a p.o.u. subordinate to $\{U_\beta \cap V_\gamma\}_{\beta,\gamma}$. $\square$

Lemma 10: If $\{U_\beta\}_{\beta \in \mathcal{B}}$ is a numerable cover of Y and $f: X \to Y$ is a continuous map, then $\{f^{-1} U_\beta\}_{\beta \in \mathcal{B}}$ is a numerable cover of X.

Pf: If $\{\varphi_\alpha\}_{\alpha \in A}$ is a p.o.u. on Y subordinate to $\{U_\beta\}_{\beta \in \mathcal{B}}$, then $\{\varphi_\alpha \circ f\}_{\beta \in \mathcal{B}}$ is a p.o.u. on X subordinate to $\{f^{-1} U_\beta\}_{\beta \in \mathcal{B}}$. $\square$

Lemma 11: If ξ and ρ are numerable v.b.'s over B, so are $\xi \oplus \rho$, $\xi \otimes \rho$, $\mathrm{Hom}(\xi, \rho)$, ξ^*, $\Lambda^k(\xi)$...

Pf: By Lemma 9, we can find a numerable cover $\{U_\beta\}_{\beta \in B}$ of B s.t. $\xi | U_\beta$ and $\rho | U_\beta$ are both trivial $\forall \beta \in \mathcal{B}$. By construction, the vector bundles $\xi \oplus \rho$, $\xi \otimes \rho$, ... are also trivial over each U_β. $\square$

Lemma 12: If $\xi \to B$ is a numerable v.b., so is $f^* \xi$ for any continuous $f: A \to B$.

Pf: Use Lemma 10 and the observation that a trivialization $\xi | U \xrightarrow{\cong} U \times \mathbb{R}^n$ induces a trivialization $f^* \xi | f^{-1} U \longrightarrow (f^{-1} U) \times \mathbb{R}^n$ (Check this!). $\square$

Here is a prototypical application of partitions of unity.

(Hermitian) (complex)

Def 13: A <u>Riemannian</u> <u>metric</u> on a real v.b. ξ is a continuous map

$$\xi \oplus \xi \xrightarrow{\langle\,,\,\rangle} \mathbb{R} \quad (\mathbb{C})$$

(a Hermitian inner product)

restricting to an inner product on all fibres.

(complex)

Prop 14: Suppose $\xi \xrightarrow{p} B$ is a numerable real v.b.

(Hermitian)

Then there exists a Riemannian metric on ξ.

Pf: Let $\{U_\beta\}_{\beta\in\mathcal{B}}$ be a numerable cover of B s.t. $\xi|U_\beta$ is trivial $\forall \beta \in \mathcal{B}$, and choose local trivializations

$$h^\beta : \xi|U_\beta \xrightarrow{\approx} U_\beta \times \mathbb{R}^n .$$

Define a Riemannian metric $\langle\,,\,\rangle_\beta$ on $\xi|U_\beta$ by setting

$$\langle v,w\rangle_\beta = \langle h^\beta_b(v), h^\beta_b(w)\rangle_{\mathbb{R}^n} \quad \text{for } v,w\in \xi_b,$$

where $\langle\,,\,\rangle_{\mathbb{R}^n}$ is the standard inner product on $\mathbb{R}^n$ and $h^\beta_b : \xi_b \xrightarrow{\approx} \mathbb{R}^n$ is the isomorphism induced by h^β. Let $\{\varphi_\beta\}_{\beta\in\mathcal{B}}$ be a p.o.u. subordinate to $\{U_\beta\}_{\beta\in\mathcal{B}}$. Define

$$\langle\,,\,\rangle : \xi \oplus \xi \longrightarrow \mathbb{R}$$

by

$$\langle v,w\rangle = \sum_{\beta\in\mathcal{B}} \varphi_\beta(p(v))\langle v,w\rangle_\beta .$$

Then $\langle\,,\,\rangle$ is a Riemannian metric on ξ

(Recall that if $\langle , \rangle_1, \dots, \langle , \rangle_k$ are inner products on a vector space V, then so is any convex combination $\sum_{i=1}^k a_i \langle , \rangle_i$, $a_1, \dots, a_k \geq 0$, $\sum_{i=1}^k a_i = 1$.)

The claim for complex v.b.'s is proved similarly. $\square$

<u>Cor 15</u>: Sp. $0 \to \xi' \xrightarrow{i} \xi \xrightarrow{p} \xi'' \to 0$ is a short exact sequence of vector bundles with ξ numerable. Then $\xi \approx \xi' \oplus \xi''$.

<u>Pf</u>: Recall that if $0 \to V' \xrightarrow{i} V \xrightarrow{p} V'' \to 0$ is a short exact sequence of vector spaces and $r : V \to V'$ is a retraction for i (i.e. $r \circ i = id_{V'}$), then $(r, p) : V \to V' \oplus V''$ is an iso

$$\begin{tikzcd}
 & & V \arrow[dll, dashed, bend right, "r"'] \arrow[dr, "p"] \arrow[dd, "(r{,}p)", "\approx"'] & & \\
0 \arrow[r] & V' \arrow[ur, "i"] \arrow[dr, "incl."'] & & V'' \arrow[r] & 0 \\
 & & V' \oplus V'' \arrow[ur, "pr"'] & &
\end{tikzcd}$$

(Check this if you haven't seen this before!)

Pick a (Riemannian / Hermitian) metric on ξ. Then the map $\xi \to \mathrm{Im}(i : \xi' \to \xi)$ given by orthogonal projection and the isomorphism $\xi' \xrightarrow{\approx} \mathrm{Im}(i : \xi' \to \xi)$ defined by i give a retraction $r : \xi \to \xi'$ for $i : \xi' \to \xi$. By the aforementioned result for vector spaces, the map

$$(r, p) : \xi \xrightarrow{\cong} \xi' \oplus \xi''$$

is an isomorphism on fibres, hence an isomorphism of vector bundles. $\square$