Lecture 5

More on paracompactness.

It is useful to know when a space is paracompact — then numerability is automatic. We will show in particular that compact Hausdorff spaces are paracompact.

Recall: A space X is normal if it is Hausdorff and any two disjoint closed subsets of X have disjoint open neighbourhoods.

Eg. Any compact Hausdorff space is normal.

(Pf: Sp. X is compact Hausdorff and $A, B \subset X$ are disjoint closed subsets. Consider first the case where A consists of a single point a. Since X is Hausdorff, for every $b \in B$, we can find disjoint open neighbourhoods U_b for a and V_b for b. Since X is compact and $B \subset X$ is closed, B is compact. Thus we can find $b_1, \ldots, b_n \in B$ s.t. $B \subset \bigcup_{i=1}^n V_{b_i}$. Then $\bigcap_{i=1}^n U_{b_i}$ and $\bigcup_{i=1}^n V_{b_i}$ are disjoint neighbourhoods of $A = \{a\}$ and B, respectively.

Consider now the general case where $A \subset X$ is an arbitrary closed subset. By the case we have already proven, for each $a \in A$ we can find disjoint neighbourhoods U_a of a and V_a of B. The subspace A is compact, so we can find $a_1, \dots, a_m \in A$ s.t. $A \subset \bigcup_{i=1}^m U_{a_i}$. Now $\bigcup_{i=1}^m U_{a_i}$ and $\bigcap_{i=1}^m V_{a_i}$ are disjoint neighbourhoods of A and B, respectively.)

<u>Thm 1</u> (Tietze extension theorem)

Sp. X is normal and $A \subset X$ is a closed subspace. Then any continuous map $f: A \to \mathbb{R}$ can be extended to a continuous map $F: X \to \mathbb{R}$. □

This is probably the most important feature of normal spaces. For proof, see eg. [James R. Munkres, Topology, 2nd edition, Thm 35.1]

<u>Rk</u>: If f is a map $A \to [a,b]$, one can choose as F a map $X \to [a,b]$.

<u>Prop 2</u>: Paracompact spaces are normal.

<u>Pf</u>: Sp. X is paracompact and $A_1, A_2 \subset X$ are disjoint closed subsets. Pick a partition of unity $\{\varphi_1, \varphi_2\}$ subordinate to the open cover $\{X \setminus A_1, X \setminus A_2\}$ of X. Then $\varphi_2 A_1 = \{1\}$ and $\varphi_1 A_2 = \{1\}$, and $\varphi_2^{-1}(3/4, 1]$ and $\varphi_1^{-1}(3/4, 1]$ are disjoint neighbourhoods of A_1 and A_2, respectively.

<u>Prop 3</u>: Compact Hausdorff spaces are paracompact.

<u>Pf</u>: Sp. X is compact Hausdorff, and let

$\{U_\beta\}_{\beta\in B}$ be an open cover of X. Since X is normal, for each $x\in X$, we can find an open neighbourhood V_x of x s.t. $\overline{V}_x \subset U_\alpha$ for some α (apply normality to $\{x\}$ and $X\setminus U_\alpha$ for some U_α containing x). Tietze $\Rightarrow$ we can find continuous fn's $\varphi_x : X \to [0,1]$ s.t. $\varphi_x(x) = 1$ and $\varphi_x(X\setminus V_x) = \{0\}$ for all $x\in X$. Now $\{\varphi_x^{-1}(0,1]\}_{x\in X}$ is an open cover of X. Pick a finite subcover $\{\varphi_{x_i}^{-1}(0,1]\}_{i=1}^{n}$. Then $\sum_{i=1}^{n} \varphi_{x_i}(x) > 0$ for all $x\in X$. Normalize by dividing each φ_{x_i} by $\sum_{i=1}^{n} \varphi_{x_i}$ $\rightsquigarrow$ p.o.u. subordinate to $\{U_\beta\}_{\beta\in B}$. $\square$

(Rk: In fact, a weak form of Tietze's theorem called Urysohn's Lemma would have sufficed for the proof.)

Further examples of paracompact spaces:

- metric spaces ([Munkres, Thm 41.4])
- CW complexes ([Hatcher, Vector bundles and K-theory, Prop. 1.20] or [Lundell and Weingram, The topology of CW complexes, Thm II.4.2])
- Spaces of the form $X = \bigcup_{i=1}^{\infty} K_i$, where $K_1 \subset K_2 \subset \cdots$ are compact Hausdorff spaces and X has the weak or direct limit topology: $A\subset X$ is closed $\Leftrightarrow$ $A\cap K_i$ is closed in K_i for all i. ([Hatcher, Prop. 1.19])

In general, subspaces of paracompact spaces need not be paracompact. However:

<u>Prop 4</u>: Any closed subspace of a paracompact space is paracompact.

(<u>Pf</u>: Sp. X is paracompact and $A \subset X$ is closed. Let $\{U_\beta\}_{\beta\in\mathcal{B}}$ be an open cover of A. For each β, pick an open $V_\beta \subset X$ s.t. $U_\beta = V_\beta \cap A$. Then $\{X \setminus A, V_\beta\}_{\beta\in\mathcal{B}}$ is an open cover of X, so we can find a p.o.u. $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ on X subordinate to it. Now $\{\varphi_\alpha|A\}_{\alpha\in\mathcal{A}}$ is a p.o.u. subordinate to $\{U_\beta\}_{\beta\in\mathcal{B}}$. $\square$)

In general, products of paracompact spaces need not be paracompact. However:

<u>Prop 5</u>: If X is paracompact and K is compact Hausdorff, then $X \times K$ is paracompact.

(<u>Pf</u>: Let us show that every open cover of $X \times K$ has a subordinate p.o.u. It suffices to consider open covers of the form $\{U_\beta \times V_\beta\}_{\beta\in\mathcal{B}}$ where $U_\beta \subset X$ and $V_\beta \subset K$ are open subsets, since every open cover of $X \times K$ has a refinement of this form. For each $x \in X$, we can find $\beta(x,1), \dots, \beta(x,n_x) \in \mathcal{B}$ s.t. the sets $U_{\beta(x,k)} \times V_{\beta(x,k)}$, $k = 1, \dots, n_x$, cover $\{x\} \times K$. Let $W_x = \bigcap_{k=1}^{n_x} U_{\beta(x,k)}$.

Then $\{W_x\}_{x\in X}$ is an open cover of X. Choose a p.o.u. $\{\varphi_i\}_{i\in I}$ on X subordinate to it. For each $i\in I$, choose an $x_i\in X$ s.t. $\mathrm{supp}(\varphi_i)\subset W_{x_i}$, and choose a p.o.u. $\{\psi_{i,j}\}_{j\in J_i}$ on K subordinate to $\{V_{\beta(x_i,k)}\}_{k=1}^{n_{x_i}}$. For $i\in I$, $j\in J_i$, let

$$\Phi_{i,j} : X\times K \longrightarrow [0,1]$$

be the fn $\Phi_{i,j}(x,y) = \varphi_i(x)\,\psi_{i,j}(y)$. We claim that $\{\Phi_{i,j}\}_{i\in I,\, j\in J_i}$ is a p.o.u. on X subordinate to $\{U_\beta\times V_\beta\}$.

First, notice that

$$\sum_{i,j}\Phi_{i,j}(x,y) = \sum_i\sum_j \varphi_i(x)\,\psi_{i,j}(x) = \sum_i \varphi_i(x) = 1$$

for all $(x,y)\in X\times K$ and that for all i,j,

$$\mathrm{supp}(\Phi_{i,j}) \subset \mathrm{supp}(\varphi_i)\times\mathrm{supp}(\psi_{i,j}) \subset W_{x_i}\times V_{\beta(x_i,k)} \subset U_{\beta(x_i,k)}\times V_{\beta(x_i,k)}$$

for some $1\le k\le n_{x_i}$. To see local finiteness, assume $(x,y)\in X\times K$. Pick a neighbourhood $U\subset X$ of x meeting $\mathrm{supp}(\varphi_i)$ for only finitely ~~many~~ many $i\in I$. Let $I_0\subset I$ be the set of these i's. For each $i\in I_0$, pick a neighbourhood $V_i\subset K$ of y s.t. V_i meets $\mathrm{supp}(\psi_{i,j})$ for only finitely many $j\in J_i$, and let $V=\bigcap_{i\in I_0} V_i$. Now $U\times V$ is a neighbourhood of (x,y) meeting $\mathrm{supp}(\Phi_{i,j})$ for only finitely many (i,j). □)

Homotopies and homotopy equivalences

Def 6: Continuous maps $f, g : X \to Y$ are homotopic, $f \simeq g$, if $\exists$ continuous $h : X \times I \to Y$ s.t. $I = [0,1]$

$$h(x,0) = f(x) \quad \text{and} \quad h(x,1) = g(x) \quad \forall x \in X.$$

Such an h is a homotopy from f to g. Write $h : f \simeq g$. If $A \subset X$, h is relative to A or rel A if

$$h(x,t) = f(x) = g(x) \qquad \forall x \in A \text{ and } t \in I.$$

A map is nullhomotopic if it is homotopic to a constant map.

Exc: The relation $\simeq$ is an equivalence relation on $\operatorname{map}(X,Y) = \{ f : X \to Y \text{ continuous} \}$.

Def 7: The class of f under $\simeq$ is the homotopy class of f, written $[f]$. Write $[X,Y]$ for the set of homotopy classes of maps $X \to Y$.

Exc: Composition of maps descends to homotopy classes in the sence that the map

$$\begin{aligned} [Y,Z] \times [X,Y] &\xrightarrow{\ \circ\ } [X,Z] \\ ([f],[g]) &\longmapsto [f \circ g] \end{aligned}$$

is well defined.

Def 8: Spaces X and Y are homotopy equivalent and have the same homotopy type if $\exists f: X \to Y$ and $g: Y \to X$ s.t. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.
Such an f is called a homotopy equivalence and g a homotopy inverse of f. X is called contractible if $X \simeq pt$.
one-point space.

Exc: X is contractible $\Leftrightarrow$ id_X is nullhomotopic.

One way to obtain homotopy equivalences is via deformation retractions.

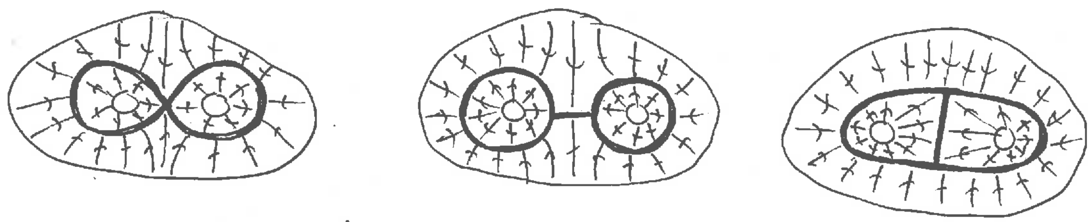
Def 9: A (strong) deformation retraction from X to a subspace $A \subset X$ is a homotopy rel A from id_X to a map taking values in A.
Call A a deformation retract of X if $\exists$ deformation retraction from X to A.

Rk: A retraction from X to A is a continuous map $r: X \to A$ such that $r(x) = x$ for all $x \in A$.
Observe that the $t=1$ part of a deformation retraction defines a retraction from X to A.

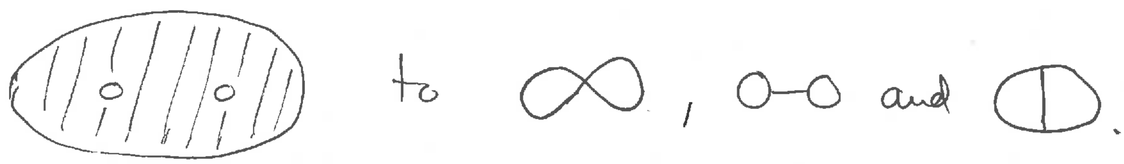
Prop 10: Sp. $A \subset X$ is a deformation retract of X.
Then the inclusion $i: A \hookrightarrow X$ is a homotopy equivalence.

Pf: Let $h: X \times I \longrightarrow X$ be a deformation retraction from X to A. Define $r: X \to A$ by $r(x) = h(x,1)$. Then $r \circ i = id_A$ and $h: id_X \simeq i \circ r$. $\square$

Eg.



Sliding along the indicated paths give deformation retractions from



Consequently, all these spaces are homotopy equivalent.

Eg. $\{0\}$ is a deformation retract of the unit n-disk $D^n = \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$; a deformation retraction is given by the map $h(x,t) = (1-t)x$. Thus D^n is contractible. Similarly, $\mathbb{R}^n$ is contractible.

Homotopy invariance of pullbacks of vector bundles

We would like to prove the following theorem:

Thm 11: Sp. ξ is a numerable v.b. over Y. Let $f, g: X \to Y$ be homotopic maps. Then the vector bundles $f^*\xi$ and $g^*\xi$ over X are isomorphic.

The proof is split into a sequence of lemmas, notably

<u>Lemma 12</u>: Let $\xi \to X \times I$ be a numerable v.b. Then $\exists$ numerable cover $\{U_\beta\}_{\beta \in B}$ of X s.t. $\xi | U_\beta \times I$ is trivial $\forall \beta \in B$.

and

<u>Lemma 13</u>: Let $\xi \to X \times I$ be a numerable v.b. Then $\exists$ Cartesian morphism $f : \xi \to \xi$ covering the map $r : X \times I \to X \times I$, $r(x,t) = (x,1)$.

iso on fibres

$$\begin{tikzcd} \xi \arrow[r, "f"] \arrow[d] & \xi \arrow[d] \\ X \times I \arrow[r, "r"] & X \times I \end{tikzcd}$$

Once Lemma 13 has been established, Thm 11 follows easily. One first observes that Lemma 13 implies

<u>Cor 14</u>: If $\xi \to X \times I$ is a numerable v.b., the v.b.'s $\xi_0 \to X$ and $\xi_1 \to X$ obtained by restricting ξ to $X \times \{0\}$ and $X \times \{1\}$, respectively, are isomorphic.

<u>Pf</u>: A Cartesian $f : \xi \to \xi$ covering $r : X \times I \to X \times I$ restricts to an iso $\xi_0 \to \xi_1$. $\square$

Thm 11 now follows by observing that if h is a homotopy from f to g, then $(h^*\xi)_0 \approx f^*\xi$ and $(h^*\xi)_1 \approx g^*\xi$.