Homotopy invariance of pullbacks of vector bundles

We would like to show

Thm 1: Sp. ξ is a numerable v.b. over Y. Let $f, g : X \to Y$ be homotopic maps. Then the v.b.'s $f^*\xi$ and $g^*\xi$ over X are isomorphic.

Before going into the proof, let me point out some important consequences of the thm.

Notation: Write $\mathrm{Vect}^n_{\mathbb{R}}(X)$ ($\mathrm{Vect}^n_{\mathbb{C}}(X)$) for the set of isomorphism classes of numerable n-dim'l real (complex) v.b.'s over X. We also write $\mathrm{Vect}^n(X)$ if we are agnostic to the choice b/w $\mathbb{R}$ and $\mathbb{C}$ or if the choice is determined by context.

Cor 2: If $f : X \to Y$ is a homotopy equivalence, then the map $f^* : \mathrm{Vect}^n(Y) \to \mathrm{Vect}^n(X)$, $[\xi] \mapsto [f^*\xi]$ is a bijection.

Pf: Let $g : Y \to X$ be a homotopy inverse for f. Then for all $[\xi] \in \mathrm{Vect}^n(Y)$ and $[\varsigma] \in \mathrm{Vect}^n(X)$ we have

$$g^* f^* \xi \approx (fg)^* \xi \approx id_Y^* \xi \approx \xi$$

and

$$f^* g^* \varsigma \approx (gf)^* \varsigma \approx id_X^* \varsigma \approx \varsigma.$$

So $g^* : \mathrm{Vect}^n(X) \to \mathrm{Vect}^n(Y)$, $[\varsigma] \mapsto [g^*\varsigma]$ is an inverse for f^*. $\square$

Cor 3: If X is contractible, all numerable v.b.'s over X are trivial.

Pf: By Cor. 2, $\mathrm{Vect}^n(X) \approx \mathrm{Vect}^n(pt) = \{[\varepsilon^n]\}$. □

Last time, we saw how Thm 1 follows from

Lemma 4: Let $\xi \to X \times I$ be a numerable v.b. Then $\exists$ Cartesian morphism $f : \xi \to \xi$ covering the map $r : X \times I \to X \times I$, $(x,t) \mapsto (x,1)$.

iso on fibres

$$\begin{tikzcd} \xi \arrow[r, "f"] \arrow[d] & \xi \arrow[d] \\ X \times I \arrow[r, "r"] & X \times I \end{tikzcd}$$

So our goal is to prove this lemma. The proof is organized as a sequence of lemmas.

Lemma 5: Sp. $\xi \to X \times [a,c]$ is a v.b. s.t. $\xi | X \times [a,b]$ and $\xi | X \times [b,c]$ are trivial for some $a < b < c$. Then ξ is trivial.

Pf: Let $h_1 : \xi | X \times [a,b] \xrightarrow{\approx} \mathbb{R}^n \times X \times [a,b]$

$h_2 : \xi | X \times [b,c] \xrightarrow{\approx} \mathbb{R}^n \times X \times [b,c]$

be trivializations. The composite

$$\mathbb{R}^n \times X \times \{b\} \xrightarrow[\approx]{(h_2|)^{-1}} \xi | X \times \{b\} \xrightarrow[\approx]{h_1|} \mathbb{R}^n \times X \times \{b\}$$

is of the form

$$(v, x, b) \longmapsto (\delta(x)v, x, b)$$

for some $\delta : X \to GL_n\mathbb{R}$. Define $\tilde{h}_2$ as the composite

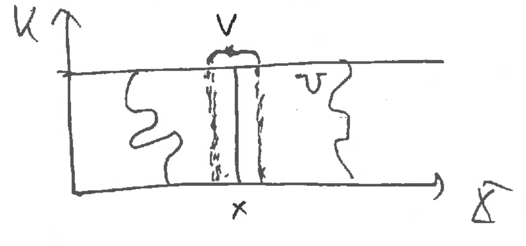
$$\tilde{h}_2 : \xi | X \times [b,c] \xrightarrow[\approx]{h_2} \mathbb{R}^n \times X \times [b,c] \xrightarrow[\approx]{} \mathbb{R}^n \times X \times [b,c]$$
$$(v, x, t) \longmapsto (\delta(x)v, x, t)$$

Then h_1 and $\tilde{h}_2$ agree on $\xi | X \times \{b\}$, and hence glue to give a trivialisation

$$\xi \longrightarrow \mathbb{R}^n \times X \times [a,c] . \quad \square$$

We recall two general results.

<u>Lemma 6</u> (Tube Lemma): Consider a product $X \times K$, where X is any space and K is compact. Let $x \in K$. Then for any neighbourhood $U \subset X \times K$ of $\{x\} \times K$, there exists a neighbourhood $V \subset X$ of x s.t. $V \times K \subset U$



<u>Pf</u>: For each $y \in K$, we can find neighbourhoods $V_y \subset X$ of x and $W_y \subset K$ of y s.t. $V_y \times W_y \subset U$. Since K is compact, there exist $y_1, \dots, y_n \in K$ s.t. $W_{y_1}, \dots, W_{y_n}$ cover K. Then $V = \bigcap_{i=1}^n V_{y_i}$ is as desired. $\square$

(The above result is useful in many arguments involving compactness.)

<u>Lemma 7</u>: Let $f: X \times K \to \mathbb{R}$ be a continuous map where K is compact. Then the map

$$g: X \to \mathbb{R} \qquad \left(\exists \min \text{ since } K \text{ is compact}\right)$$
$$x \mapsto \min_{y \in K} f(x,y)$$

is continuous.

<u>Pf</u>: Let $x_0 \in X$. We will show that g is continuous at x_0. Let $\varepsilon > 0$. Pick $y_0 \in K$ s.t. $g(x_0) = f(x_0, y_0)$. Choose neighbourhoods $U \subset X$ of x_0 and $V \subset K$ of y_0 s.t. $f(x,y) < g(x_0) + \varepsilon$ $\forall (x,y) \in U \times V$. Then $g(x) < g(x_0) + \varepsilon$ $\forall x \in U$. Observe that $f^{-1}((g(x_0) - \varepsilon, \infty)) \subset X \times K$ is a neighbourhood of $\{x_0\} \times K$. By the Tube lemma, we can find a neighbourhood W of x_0 s.t. $W \times K \subset f^{-1}((g(x_0) - \varepsilon, \infty))$. For $x \in W$ we have $f(x,y) > g(x_0) - \varepsilon$ for all $y \in K$, and hence $g(x) > g(x_0) - \varepsilon$. Thus $U \cap W$ is a neighbourhood of x_0 s.t. $g(U \cap W) \subset (g(x_0) - \varepsilon, g(x_0) + \varepsilon)$. □

The following lemma is the most difficult step in the proof of Thm 1.

<u>Lemma 8</u>: Let $\xi \to X \times I$ be a numerable v.b. Then $\exists$ numerable cover $\{U_\beta\}_{\beta \in B}$ of X s.t. $\xi | U_\beta \times I$ is trivial $\forall \beta \in B$.

Pf: Let $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ be a numerable cover of $X \times I$ s.t. $\xi|V_\alpha$ is trivial $\forall \alpha\in\mathcal{A}$, and let $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ be a p.o.u. subordinate to it. Let

$$\mathcal{B} = \{(\beta(1),\dots,\beta(r)) \mid r\geq 1, \beta(i)\in\mathcal{A}\ \forall i\}$$

and write

$$|\beta| = r \quad \text{if } \beta = (\beta(1),\dots,\beta(r))\in\mathcal{B} \quad (|\beta| = \underline{\text{length}} \text{ of } \beta).$$

Define

$$v_\beta(x) = \min_{1\leq q\leq|\beta|} \ \min_{t\in[\frac{q-1}{|\beta|},\frac{q}{|\beta|}]} \varphi_{\beta(q)}(x,t).$$

Then Lemma 7 implies that $v_\beta : X \to [0,1]$ is continuous. Moreover,

$$v_\beta(x) > 0 \iff \{x\}\times\left[\tfrac{q-1}{|\beta|},\tfrac{q}{|\beta|}\right] \subset \varphi_{\beta(q)}^{-1}(0,1] \ \forall 1\leq q\leq |\beta|.$$

So trivializations of ξ on $V_{\beta(q)}$'s and Lemma 5 give a trivialization of ξ on $v_\beta^{-1}(0,1]\times I$. Thus it is enough to show that the sets $U_\beta = v_\beta^{-1}(0,1] \subset X$ give a numerable cover of X.

<u>$\{U_\beta\}_{\beta\in\mathcal{B}}$ is a cover:</u>

Sp. $x\in X$. The open sets $\varphi_\alpha^{-1}(0,1]$, $\alpha\in\mathcal{A}$ cover $\{x\}\times I$, so by the compactness of I, picking $r\geq 1$ large enough, we can find $\beta(1),\dots,\beta(r)\in\mathcal{A}$ s.t.

$$\{x\}\times\left[\tfrac{q-1}{r},\tfrac{q}{r}\right] \subset \varphi_{\beta(q)}^{-1}(0,1] \quad \forall 1\leq q\leq r.$$

Then $v_\beta(x) > 0$ for $\beta = (\beta(1),\dots,\beta(r))$, i.e. $x\in U_\beta$.

$\{v_\beta\}_{\beta \in B}$ is numerable

Step 1: $\{\mathrm{supp}(v_\beta)\}_{\beta\in B, |\beta|\leq m}$ is locally finite $\forall m \geq 1$:

every $x \in X$ has a neighbourhood meeting only finitely many of the sets

Sp. $x \in X$. For each $t \in I$, we can find a neighbourhood $W_t \subset X \times I$ of (x,t) meeting $\mathrm{supp}(\varphi_\alpha)$ for only finitely many $\alpha \in A$. Since I is compact, $\exists\, t_1, \dots, t_n \in I$ s.t. $\{x\} \times I \subset \bigcup_{i=1}^n W_{t_i}$. By the Tube lemma, we can find a neighbourhood $N \subset X$ of x s.t. $N \times I \subset \bigcup_{i=1}^n W_{t_i}$. Then $N \times I$ meets $\mathrm{supp}(\varphi_\alpha)$ for only finitely many $\alpha \in A$. Therefore, for any fixed $m \geq 1$, N meets $\mathrm{supp}(v_\beta)$ for only finitely many $\beta \in B$ with $|\beta| \leq m$.

Step 2: Construct a p.o.u. subordinate to $\{v_\beta\}_{\beta\in B}$.

For $r \geq 1$, define $w_r : X \to [0,\infty)$ by

$$w_r(x) = \sum_{\substack{\beta \in B \\ |\beta| < r}} v_\beta(x).$$

(Step 1 $\Rightarrow$ the sum makes sense & is continuous.)

For $\beta \in B$, set

$$u_\beta(x) = \max\left(0, v_\beta(x) - |\beta|\, w_{|\beta|}(x)\right).$$

For every $x \in X$, $\exists\, \beta_0 \in B$ with $|\beta_0|$ minimal s.t. $v_{\beta_0}(x) > 0$. Then $w_{\beta_0}(x) = 0$, and hence $u_{\beta_0}(x) = v_{\beta_0}(x) > 0$. Pick $m > |\beta_0|$ s.t. $v_{\beta_0}(x) > 1/m$. Then $w_m(x) > 1/m$, and hence $w_m(y) > 1/m$ for all y in a neighbourhood $W \subset X$ of x. For all $s \geq m$, we then have

$$s w_s(y) \geq m w_m(y) > 1 \quad \text{for } y \in W,$$

and hence $u_\beta(y) = 0$ for $y \in W$ for all $\beta \in \mathcal{B}, |\beta| \geq m$.
Since $\mathrm{supp}(u_\beta) \subset \mathrm{supp}(v_\beta)$ for all $\beta \in \mathcal{B}$, from Step 1 we can conclude that $\{\mathrm{supp}(u_\beta)\}_{\beta\in\mathcal{B}}$ is locally finite.

Problem: we have $u_\beta^{-1}(0,1] \subset v_\beta^{-1}(0,1] = U_\beta$, but this is not sufficient to ensure that $\mathrm{supp}(u_\beta) = \overline{u_\beta^{-1}(0,1]} \subset U_\beta$. So we would like to construct a new family of functions with slightly smaller supports (while retaining the property that for every $x \in X$, at least one of the fn's is positive at x). We can do this as follows:

Define $u'_\beta : X \to [0,1]$ by

$$u'_\beta(x) = \frac{u_\beta(x)}{\max_{\gamma\in\mathcal{B}} u_\gamma(x)} .$$

Then $\mathrm{supp}(u'_\beta) = \mathrm{supp}(u_\beta)$, and $\forall x \in X\ \exists \beta \in \mathcal{B}$ s.t. $u'_\beta(x) = 1$. Define $u''_\beta : X \to [0,1]$ by

$$u''_\beta(x) = \max\left(u'_\beta(x) - \tfrac{1}{2}, 0\right).$$

Then $\forall x \in X\ \exists \beta \in \mathcal{B}$ s.t. $u''_\beta(x) = \tfrac{1}{2} > 0$. Moreover, for all $\beta \in \mathcal{B}$ we have

$$\begin{aligned}
\mathrm{supp}(u''_\beta) &= \overline{(u''_\beta)^{-1}(0,1]} \\
&= \overline{(u'_\beta)^{-1}(\tfrac{1}{2},1]} \\
&\subset (u'_\beta)^{-1}[\tfrac{1}{2},1] \\
&\subset (u'_\beta)^{-1}(0,1] \\
&= u_\beta^{-1}(0,1] \\
&\subset v_\beta^{-1}(0,1] = U_\beta .
\end{aligned}$$

Now the functions $\psi_\beta : X \to [0,1]$,

$$\psi_\beta(x) = \frac{u''_\beta(x)}{\sum_{\gamma \in \mathcal{B}} u''_\gamma(x)}$$

give a p.o.u. subordinate to $\{V_\beta\}_{\beta \in \mathcal{B}}$.

<u>Next time</u>: proof of Lemma 4.