Homotopy invariance of pullbacks of vector bundles (cont.)

Recall: We are trying to show

Thm 1: Sp. $\xi \to Y$ is a numerable v.b. Let $f, g : X \to Y$ be homotopic maps. Then the v.b.'s $f^*\xi$ and $g^*\xi$ over X are isomorphic.

We have reduced the thm to showing

Lemma 2: Let $\xi \to X \times I$ be a numerable v.b. Then $\exists$ Cartesian morphism $f : \xi \to \xi$ covering the map $r : X \times I \to X \times I$, $(x,t) \mapsto (x,1)$.

iso on fibres

$$\begin{tikzcd} \xi \arrow[r, "f"] \arrow[d] & \xi \arrow[d] \\ X \times I \arrow[r, "r"] & X \times I \end{tikzcd}$$

Last time, we proved

Lemma 3: Let $\xi \to X \times I$ be a numerable v.b. Then $\exists$ numerable cover $\{U_\beta\}_{\beta \in B}$ of X s.t. $\xi | U_\beta \times I$ is trivial $\forall \beta \in B$. □

Pf of Lemma 2: Let $\{U_\beta\}$ be a cover of X of the type afforded by Lemma 3, and choose local trivializations $h_\beta : \xi | U_\beta \xrightarrow{\approx} U_\beta \times I \times \mathbb{R}^n$ for $\beta \in B$ and a p.o.u. $\{\varphi_\beta\}_{\beta \in B}$ subordinate to $\{U_\beta\}_{\beta \in B}$.

For $\beta \in \mathcal{B}$, define a Cartesian morphism

$$\begin{tikzcd} \xi \arrow[r, "f_\beta"] \arrow[d] & \xi \arrow[d] \\ X \times I \arrow[r, "r_\beta"] & X \times I \end{tikzcd}$$

as follows:

* r_β is the map $r_\beta(x,t) = (x, \min(1, t + \varphi_\beta(x)))$.
* Over $U_\beta \times I$, f_β is the map

$$\xi | U_\beta \times I \longrightarrow \xi | U_\beta \times I$$

which, under h_β, corresponds to the map

$$\begin{aligned} U_\beta \times I \times \mathbb{R}^n &\longrightarrow U_\beta \times I \times \mathbb{R}^n \\ (x, t, v) &\longmapsto (r_\beta(x,t), v) \end{aligned}$$

* Outside $\mathrm{supp}(\varphi_\beta)$, f_β is the identity.

Check: this gives a well-defined Cartesian morphism covering r_β.

Pick a total order on $\mathcal{B}$. Using the order, we can make sense of the composite

$$\bigcirc_{\beta \in \mathcal{B}} f_\beta : \xi \longrightarrow \xi$$

as follows: For each $x \in X$, we can find a neighbourhood $V \subset X$ of x s.t. V meets $\mathrm{supp}(\varphi_\beta)$ for only finitely many $\beta \in \mathcal{B}$, say for $\beta_1, \dots, \beta_k \in \mathcal{B}$, where $\beta_1 < \beta_2 < \dots < \beta_k$.

We let

$$\left(\underset{\beta\in B}{\circ} f_\beta\right)(v) = (f_{\beta_1}\circ\cdots\circ f_{\beta_k})(v) \quad \text{for } v\in\xi|V\times I.$$

Check: this gives a well-defined map $\underset{\beta\in B}{\circ} f_\beta : \xi\to\xi$.

Now $f = \underset{\beta\in B}{\bigcirc} f_\beta$ is as desired. $\square$

This concludes the proof of Thm 1.

Next goal:

Thm 4: There is a bijection

$$\mathrm{Vect}^k_{\mathbb{F}}(X) \xleftarrow{\approx} [X, Gr_k(\mathbb{F}^\infty)].$$
$$[f^*\gamma^k] \longleftarrow\!\shortmid [f]$$

Here $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, and the space $Gr_k(\mathbb{F}^\infty)$ (the Grassmannian of k-planes in $\mathbb{F}^\infty$) and the vector bundle $\gamma^k\to Gr_k(\mathbb{F}^\infty)$ are defined below. Thm 1 $\Rightarrow$ the above map is well-defined.

Rk: Everything we have done so far in this course has worked equally well for both real and complex v.b.'s, even though in the notation we have occasionally assumed the real case. From now on, let us adopt the convention that $\mathbb{F}$ stands for either $\mathbb{R}$ or $\mathbb{C}$.

Grassmannians and their canonical vector bundles

Let

$$\mathbb{F}^\infty = \{(a_1, a_2, \ldots) \mid a_i \in \mathbb{F}\ \forall i \ \& \ a_i \neq 0 \text{ for only finitely many } i\}.$$

This is an $\mathbb{F}$-vector space. For $0 \leq n \leq \infty$, $\mathbb{R}^n$ has the inner product

$$\langle (a_i)_{i=1}^n, (b_i)_{i=1}^n \rangle = \sum_{i=1}^n a_i b_i$$

and $\mathbb{C}^n$ the Hermitian inner product

$$\langle (a_i)_{i=1}^n, (b_i)_{i=1}^n \rangle = \sum_{i=1}^n a_i \overline{b_i}.$$

For $k \leq n \leq \infty$, define

$$Gr_k(\mathbb{F}^n) = \{k\text{-dim'l vector subspaces of } \mathbb{F}^n\},$$

the Grassmannian of k-planes in $\mathbb{F}^n$, and

$$\gamma^k = \gamma^k(\mathbb{F}^n) = \{(V, v) \in Gr_k(\mathbb{F}^n) \times \mathbb{F}^n \mid v \in V\},$$

$$\begin{aligned} \gamma^k(\mathbb{F}^n) &\xrightarrow{p} Gr_k(\mathbb{F}^n) \\ (V, v) &\longmapsto V \end{aligned}$$

(so $p^{-1}(V) = \{V\} \times V$, "the vector space V itself")

Goal: Give $Gr_k(\mathbb{F}^n)$ and $\gamma^k(\mathbb{F}^n)$ topologies making $p: \gamma^k(\mathbb{F}^n) \longrightarrow Gr_k(\mathbb{F}^n)$ into a numerable $\mathbb{F}$-v.b., the canonical / universal v.b. over $Gr_k(\mathbb{F}^n)$.

Case $n<\infty$:

Let
$$V_k(\mathbb{F}^n) = \{(v_1,\dots,v_k)\in(\mathbb{F}^n)^k \mid v_1,\dots,v_k \text{ linearly independent}\}.$$
(k-frames)

(Such $V_k(\mathbb{F}^n)$'s are called Stiefel manifolds.)

Then $V_k(\mathbb{F}^n)\subset(\mathbb{F}^n)^k$ is an open subspace.

We have a surjection
$$q: V_k(\mathbb{F}^n) \longrightarrow Gr_k(\mathbb{F}^n)$$
$$(v_1,\dots,v_k)\longmapsto \operatorname{span}(v_1,\dots,v_k)$$
and we give $Gr_k(\mathbb{F}^n)$ the quotient topology from $V_k(\mathbb{F}^n)$. Inside $V_k(\mathbb{F}^n)$, we have the subspace
$$V_k^o(\mathbb{F}^n) = \{(v_1,\dots,v_k)\in V_k(\mathbb{F}^n) \mid \langle v_i,v_j\rangle=\delta_{ij}\}.$$
(orthonormal k-frames)

The restriction $q| : V_k^o(\mathbb{F}^n)\longrightarrow Gr_k(\mathbb{F}^n)$ is also a surjection, and diagram
$$\begin{tikzcd}
V_k^o(\mathbb{F}^n) \arrow[r, hook] \arrow[dr, "q|"'] & V_k(\mathbb{F}^n) \arrow[r, "\text{Gram-Schmidt}"] \arrow[d, "q"] & V_k^o(\mathbb{F}^n) \arrow[dl, "q|"] \\
& Gr_k(\mathbb{F}^n) &
\end{tikzcd}$$

$\rightsquigarrow$ the quotient topologies on $Gr_k(\mathbb{F}^n)$ induced by q and $q|V_k^o(\mathbb{F}^n)$ agree.

$V_k^\circ(\mathbb{F}^n)$ is a closed subset of $S(\mathbb{F}^n)^k$, where $S(\mathbb{F}^n)$ denotes the unit sphere in $\mathbb{F}^n$. Thus $V_k^\circ(\mathbb{F}^n)$ and hence $Gr_k(\mathbb{F}^n) = q\, V_k^\circ(\mathbb{F}^n)$ are compact. The map

$$\begin{array}{ccc} Gr_k(\mathbb{F}^n) & \xrightarrow{\quad \wp \quad} & End(\mathbb{F}^n) \approx \mathbb{F}^{n^2} \\ V & \longmapsto & \left(\begin{array}{c}\text{orthogonal projection} \\ \text{of } \mathbb{F}^n \text{ onto } V\end{array}\right) \end{array}$$

is injective, and it is continuous since the composite

$$\begin{array}{ccccc} V_k^\circ(\mathbb{F}^n) & \xrightarrow{\ q\,|\ } & Gr_k(\mathbb{F}^n) & \xrightarrow{\quad \wp \quad} & End(\mathbb{F}^n) \\ (v_1, \ldots, v_k) & \longmapsto & & & \left(v \mapsto \sum_{i=1}^{k} \langle v, v_i \rangle v_i\right) \end{array}$$

is. Since $Gr_k(\mathbb{F}^n)$ is compact and $End(\mathbb{F}^n)$ is Hausdorff, the map $\wp$ is an embedding. In particular, $Gr_k(\mathbb{F}^n)$ is Hausdorff.

Give $\gamma^k(\mathbb{F}^n) \subset Gr_k(\mathbb{F}^n) \times \mathbb{F}^n$ the subspace topology. Then the map $p : \gamma^k(\mathbb{F}^n) \to Gr_k(\mathbb{F}^n)$ is continuous, and we have already seen that the fibres of p are vector spaces. To see that $p : \gamma^k(\mathbb{F}^n) \to Gr_k(\mathbb{F}^n)$ is a v.b., it remains to check local triviality. Given $V \in Gr_k(\mathbb{F}^n)$, write $\wp_V : \mathbb{F}^n \to V$ for the orthogonal projection. Then

$$U_V = \{ W \in Gr_k(\mathbb{F}^n) \mid \wp_V(W) = V \}$$

is an open neighbourhood of V, and

$$\gamma^k(\mathbb{F}^n)|U_V \xrightarrow{\approx} U_V \times V$$
$$(W,w) \longmapsto (W, p_V(w))$$

is a local trivialization. Thus $\gamma^k(\mathbb{F}^n) \to Gr_k(\mathbb{F}^n)$ is a v.b. Since $Gr_k(\mathbb{F}^n)$ is compact Hausdorff, $\gamma^k(\mathbb{F}^n)$ is automatically numerable.

Case $n=\infty$:

Def 5: Let $X_1 \subset X_2 \subset \cdots$ be an ascending sequence of spaces (each with the subspace topology from the next), and let $X = \bigcup_{n=1}^{\infty} X_n$. The weak or direct limit topology on X is the topology in which $U \subset X$ is open if and only if $U \cap X_n$ is open in X_n $\forall n$. (Equivalently, $C \subset X$ is closed in X if and only if $C \cap X_n$ is closed in X_n $\forall n$.)
Check: this is a topology!

Exc: The subspace topology $X_n \subset X$ inherits from the direct limit topology on X agrees with the original topology on X_n.

Exc: For any space Y, a function $f: X \to Y$ is continuous w.r.t. the direct limit topology if and only if $f|: X_n \to Y$ is continuous for all n.

The inclusions $\mathbb{F}^r \subset \mathbb{F}^{r+1}$, $(a_1,\dots,a_r) \mapsto (a_1,\dots,a_r,0)$ induce a sequence of inclusions

$$Gr_k(\mathbb{F}^k) \subset Gr_k(\mathbb{F}^{k+1}) \subset \cdots,$$

and we have $\bigcup_{n \geq k} Gr_k(\mathbb{F}^n) = Gr_k(\mathbb{F}^\infty)$.

Give $Gr_k(\mathbb{F}^\infty)$ the direct limit topology. Give $\mathbb{F}^\infty$ the direct limit topology from

$$\mathbb{F}^1 \subset \mathbb{F}^2 \subset \cdots,$$

the space $Gr_k(\mathbb{F}^\infty) \times \mathbb{F}^\infty$ the product topology, and $\gamma^k(\mathbb{F}^\infty) \subset Gr_k(\mathbb{F}^\infty) \times \mathbb{F}^\infty$ the subspace topology. Then $p : \gamma^k(\mathbb{F}^\infty) \to Gr_k(\mathbb{F}^\infty)$ is continuous.

Next time: local triviality of $\gamma^k(\mathbb{F}^\infty)$.