Homotopy classification of vector bundles (cont.)

Recall: We are trying to show

Thm 1: For any space X, the map

$$\begin{aligned} [X, Gr_k(\mathbb{F}^\infty)] &\longrightarrow Vect^k(X) \\ [f] &\longmapsto [f^* \gamma^k] \end{aligned}$$

is a bijection.

Last time, we defined

Def 2: A Gauss map for a v.b. $\xi \to X$ is a map $\xi \to \mathbb{F}^n$ $(n \le \infty)$ s.t. $\varphi| : \xi_x \to \mathbb{F}^n$ is a linear mono $\forall x \in X$.

and showed

Prop 3: Sp. $\xi \to X$ is a k-dim'l v.b. Then

$$\{\text{Gauss maps } \xi \to \mathbb{F}^n\} \overset{\approx}{\longleftrightarrow} \left\{ \text{Cartesian morphisms } \begin{tikzcd} \xi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma^k(\mathbb{F}^n) \arrow[d] \\ X \arrow[r] & Gr_k(\mathbb{F}^n) \end{tikzcd} \right\}$$

Using Prop. 3, we showed that Thm 1 follows from

Thm 4: Sp. $\xi \to X$ is a numerable k-dim'l v.b. Then

(i) $\exists$ Gauss map $\varphi : \xi \to \mathbb{F}^\infty$

(ii) Any two Gauss maps $\xi \to \mathbb{F}^\infty$ for ξ are homotopic through Gauss maps.

Finally, we proved part (ii) of Thm 4.

To prove part (i), we need

<u>Lemma 5</u>: Let $\{U_\beta\}_{\beta\in B}$ be a numerable cover of a space X. Then $\exists$ numerable cover $\{V_i\}_{i=1}^{\infty}$ of X s.t. each V_i is a disjoint union of open sets each of which is contained in some U_β.

<u>Pf</u>: Pick a p.o.u. $\{\varphi_\beta\}_{\beta\in B}$ subordinate to $\{U_\beta\}_{\beta\in B}$. For $\emptyset \neq S \subset B$ finite, let

$$W(S) = \{x\in X \mid \varphi_\beta(x) > \varphi_\gamma(x) \ \forall \beta\in S, \gamma\in B\setminus S\}$$

and define

$$u_S : X \longrightarrow [0,1], \quad u_S(x) = \max\Big(0, \min_{\beta\in S, \gamma\in B\setminus S}(\varphi_\beta(x)-\varphi_\gamma(x))\Big)$$

Local finiteness of $\{\varphi_\beta\}$ $\Rightarrow$ in the definition of u_S, near each $x\in X$, it is enough to consider γ's in a finite subset of $B\setminus S$. It follows that the def'n of u_S makes sense, and that u_S is continuous. Thus $W(S) = u_S^{-1}(0,1]$ is open. Notice that $W(S) \subset \varphi_\beta^{-1}(0,1] \subset U_\beta$ for all $\beta\in S$, and that the sets $W(S)$ cover X: for every $x\in X$, we have $x\in W(S)$ for $S = \{\beta\in B \mid \varphi_\beta(x)>0\}$. Let

$$V_i = \bigcup_{|S|=i} W(S), \qquad w_i = \sum_{|S|=i} u_S.$$

The union is clearly disjoint, and $w_i : X \to [0,1]$ is continuous w/ $w_i^{-1}(0,1] = V_i$.

The cover $\{V_i\}_{i\geq 1}^{\infty}$ is locally finite, for if V is a neighbourhood of $x \in X$ meeting $\mathrm{supp}(\varphi_\beta)$ for only finitely many, say n, elements $\beta \in B$, then $V \cap V_i = \emptyset$ $\forall i > n$. We can now use the maps w_i to construct a p.o.u. subordinate to $\{V_i\}_{i\geq 1}^{\infty}$ by a similar method we used in the pf of lemma VI.8. Let

$$w_i' : X \to [0,1], \quad w_i'(x) = \frac{w_i(x)}{\max_{m\geq 1} w_m(x)}.$$

Then $(w_i')^{-1}(0,1] = w_i^{-1}(0,1]$ $\forall i$, and $\forall x \in X\ \exists i$ s.t. $w_i'(x) = 1$. Let

$$w_i'' : X \to [0,1], \quad w_i''(x) = \max(0, w_i'(x) - 1/2).$$

Then $\forall x \in X\ \exists i$ s.t. $w_i''(x) = 1/2 > 0$, and

$$\begin{aligned}
\mathrm{supp}(w_i'') &= \overline{(w_i'')^{-1}(0,1]} \\
&= \overline{(w_i')^{-1}(1/2,1]} \\
&\subset (w_i')^{-1}[1/2,1] \\
&\subset (w_i')^{-1}(0,1] \\
&= w_i^{-1}(0,1] \\
&= V_i
\end{aligned}$$

for all i. Normalize by dividing each w_i'' by $\sum_{m\geq 1} w_m$ $\rightsquigarrow$ p.o.u. subordinate to $\{V_i\}_{i\geq 1}^{\infty}$. $\square$

Pf of Thm 4.(i):

By Lemma 5, we can find a countable numerable cover $\{V_i\}_{i\geq 1}^{\infty}$ of X s.t. $\xi|V_i$ is trivial $\forall i$. Pick trivializations

$$h_i : \xi|V_i \xrightarrow{\approx} V_i \times \mathbb{F}^k$$

and a p.o.u. $\{\varphi_i\}_{i\geq 1}^{\infty}$ subordinate to $\{V_i\}_{i\geq 1}^{\infty}$. Let $h_i' : \xi|V_i \longrightarrow \mathbb{F}^k$ be the composite

$$\xi|V_i \xrightarrow[\approx]{h_i} V_i \times \mathbb{F}^k \xrightarrow{pr} \mathbb{F}^k$$

and define

$$\varphi_i : \xi \longrightarrow \mathbb{F}^k, \quad \varphi(v) = \varphi_i(p_\xi(v))\, h_i'(v)$$

where $p_\xi : \xi \to X$ is the projection of ξ.
Then $\varphi_i| : \xi_x \to \mathbb{F}^k$ is linear $\forall x \in X$, an iso for $x \in \varphi_i^{-1}(0,1]$, and vanishes outside $\varphi_i^{-1}(0,1]$.
Now

$$\begin{aligned} \varphi : \xi &\longrightarrow (\mathbb{F}^k \oplus \mathbb{F}^k \oplus \cdots) \approx \mathbb{F}^\infty \\ v &\longmapsto (\varphi_1(v), \varphi_2(v), \dots) \end{aligned}$$

is a Gauss map. (To see that φ is continuous, observe that for every $v \in \xi$, $\exists$ neighbourhood $U \subset X$ of $p_\xi(v)$ s.t. $\varphi|(\xi|U)$ factors as

$$\xi|U \longrightarrow \underbrace{\mathbb{F}^k \oplus \cdots \oplus \mathbb{F}^k}_{n} \hookrightarrow (\mathbb{F}^k \oplus \mathbb{F}^k \oplus \cdots)$$

for some n.) $\square$

Existence of complements

Question: Sp. $\xi \to X$ is a v.b. When does there exist a v.b. $\zeta \to X$ s.t. $\xi \oplus \zeta$ is trivial?

Def 6: Call such a v.b. $\zeta \to X$ a complement for ξ.

Eg 7: For $n < \infty$, the v.b.

$$\gamma^k(\mathbb{F}^n)^\perp = \{(V,v) \in Gr_k(\mathbb{F}^n) \times \mathbb{F}^n \mid v \perp V\} \longrightarrow Gr_k(\mathbb{F}^n)$$
$$(V,v) \longmapsto V$$

is a complement for $\gamma^k(\mathbb{F}^n)$.

Def 8: Call a v.b. $\xi \to X$ numerable of finite type if $\exists$ finite numerable cover $\{U_\beta\}_{\beta \in B}$ s.t. $\xi|U_\beta$ is trivial $\forall \beta \in B$.

Eg 9: Examples of such v.b.'s:

- any v.b. over a compact Hausdorff space
- any pullback of a numerable v.b. of finite type.

Prop 10: Let $\xi \to X$ be a v.b. Then the following are equivalent:

(i) ξ is numerable of finite type

(ii) $\exists$ Gauss map $\xi \to \mathbb{F}^n$ for some $n < \infty$

(iii) ξ admits a complement.

Pf: (i) $\Rightarrow$ (ii): Similar to the pf of Thm 4.(i).

(iii) $\Rightarrow$ (ii): If $\zeta \to X$ is a complement for ξ and $h: \xi \oplus \zeta \longrightarrow X \times \mathbb{F}^n$ is a trivialization, then

$$\xi \hookrightarrow \xi \oplus \zeta \xrightarrow[\approx]{h} X \times \mathbb{F}^n \xrightarrow{pr} \mathbb{F}^n$$

is a Gauss map.

(ii) $\Rightarrow$ (i) & (iii): By Prop. 3, $\xi \approx f^* \gamma^k(\mathbb{F}^n)$ for some $f: X \longrightarrow Gr_k(\mathbb{F}^n)$, so by Eg. 9, ξ is numerable of finite type. Moreover, $f^*(\gamma(\mathbb{F}^n)^\perp)$ provides a complement for ξ. $\square$

<u>Cor 11</u>: Any v.b. over a compact Hausdorff space admits a complement. $\square$

The clutching construction

(There are various settings and generalities in which the construction could be made. For us, the setting of compact Hausdorff spaces should be enough.)

Sp. X is a compact Hausdorff space and $X_1, X_2 \subset X$ are closed subspaces s.t. $X = X_1 \cup X_2$. Let $A = X_1 \cap X_2$. Given v.b.'s $\xi_1 \to X_1$, $\xi_2 \to X_2$ and an isomorphism

$$\varphi : \xi_1 | A \xrightarrow{\approx} \xi_2 | A$$

(<u>clutching data</u>; φ is called a <u>clutching isomorphism</u>), we can construct a v.b.

$$\xi_1 \cup_\varphi \xi_2 \longrightarrow X$$

by the following gluing/<u>clutching construction</u>:

- As a space $\xi_1 \cup_\varphi \xi_2 = \xi_1 \amalg \xi_2 / \sim$ where $\sim$ identifies each $v \in \xi_1 | A$ with $\varphi(v) \in \xi_2 | A$.
- The projection $\xi_1 \cup_\varphi \xi_2 \to X$ is the map induced by $\xi_1 \to X_1 \hookrightarrow X$ and $\xi_2 \to X_2 \hookrightarrow X$.
- The fibres of $\xi_1 \cup_\varphi \xi_2 \to X$ have the vector space str. from $\xi_1 \to X_1$ and $\xi_2 \to X_2$.

<u>Lemma 12</u>: $\xi_1 \cup_\varphi \xi_2 \longrightarrow X$ is locally trivial.

Pf: $\xi_1 \cup_\varphi \xi_2 |(X\setminus A) \approx \xi_1|(X_1\setminus A) \amalg \xi_2|(X_2\setminus A)$, so $\xi_1\cup_\varphi \xi_2 \longrightarrow X$ is locally trivial near all $x\in X\setminus A$. So $a\in A$. Pick neighbourhoods $U_i \subset X_i$ of a and trivializations $h_i : \xi_i|U_i \xrightarrow{\approx} U_i\times \mathbb{F}^n$, $i=1,2$. We will construct a modification of h_2 which will match up, under φ, with h_1 near a. Pick a neighbourhood $V\subset A$ of a s.t. $\bar V \subset U_1\cap U_2$. Over $\bar V$, the difference of h_1 and h_2 under φ is described by the map

$$\bar V\times \mathbb{F}^n \xrightarrow[\approx]{h_2^{-1}|} \xi_2|\bar V \xrightarrow[\approx]{\varphi^{-1}|} \xi_1|\bar V \xrightarrow[\approx]{h_1} \bar V\times\mathbb{F}^n$$

which has the form

$$(x,v) \longmapsto (x, f(x)v)$$

for some continuous $f:\bar V \longrightarrow GL_n(\mathbb{F})\subset \mathbb{F}^{n^2}$.

By the Tietze extension theorem, we can extend f to a map $F: \bar V \longrightarrow \mathbb{F}^{n^2}$. Since $GL_n(\mathbb{F})\subset \mathbb{F}^{n^2}$ is open, there exists a neighbourhood $W\subset X_2$ of $\bar V$ s.t. $F|W$ takes values in $GL_n(\mathbb{F})$. Pick neighbourhoods $V_1\subset X_1$ and $V_2\subset X_2$ of a s.t. $V_1\subset U_1$, $V_2\subset U_2\cap W$, and $V_1\cap A = V_2\cap A = V$.

Now the trivializations

$$\xi_1|V_1 \xrightarrow[\approx]{h_1|} V_1\times\mathbb{F}^n$$

and

$$\xi_2|V_2 \xrightarrow[\approx]{h_2|} V_2\times \mathbb{F}^n \xrightarrow{\approx} V_2\times\mathbb{F}^n$$
$$(x,v) \longmapsto (x, F(x)v)$$

match under φ to define a trivialization

$$(\xi_1\cup_\varphi \xi_2)|V \xrightarrow{\approx} V\times\mathbb{F}^n. \quad \square$$

<u>Prop 12</u>: If $\varphi_0, \varphi_1 : \xi_1|A \xrightarrow{\approx} \xi_2|A$ are homotopic through isomorphisms $\xi_1|A \xrightarrow{\approx} \xi_2|A$, then $\xi_1 \cup_{\varphi_0} \xi_2 \approx \xi_1 \cup_{\varphi_1} \xi_2$.

<u>Pf</u>: Say $(\xi_1|A) \times I \longrightarrow \xi_2|A$, $(v,t) \mapsto \varphi_t(v)$ is a homotopy from φ_0 to φ_1 s.t.

$$\varphi_t : \xi_1|A \longrightarrow \xi_2|A$$

is an iso $\forall t \in I$. Now the clutching construction for the product v.b.'s $\xi_i \times I \to X_i \times I$, $i=1,2$, and the iso

$$\begin{aligned}\Phi : (\xi_1 \times I)|A \times I &\xrightarrow{\approx} (\xi_2 \times I)|A \times I\\ (v,t) &\longmapsto (\varphi_t(v), t)\end{aligned}$$

gives us a v.b. $(\xi_1 \times I) \cup_\Phi (\xi_2 \times I) \to X \times I$ s.t.

$$i_t^*\big((\xi_1 \times I) \cup_\Phi (\xi_2 \times I)\big) \approx \xi_1 \cup_{\varphi_t} \xi_2$$

for all $t \in I$. Here $i_t : X \to X \times I$ is the map $i_t(x) = (x,t)$. Since $i_0, i_1 : X \to X \times I$ are homotopic, the claim follows. □

<u>Rk 13</u>: In the important special case where $\xi_i = X_i \times \mathbb{F}^n$, $i=1,2$, an iso $\xi_1|A \xrightarrow{\approx} \xi_2|A$ amounts to a map $A \longrightarrow GL_n(\mathbb{F})$, and Prop 12 implies that the clutching constructions associated to homotopic maps $A \longrightarrow GL_n(\mathbb{F})$ are isomorphic.