The clutching construction (cont.)

Recall our setting from last time:

X compact Hausdorff

$X_1, X_2 \subset X$ closed subspaces

$X_1 \cup X_2 = X$

$A := X_1 \cap X_2$.

Given v.b.'s $\xi_i \to X_i$, $i=1,2$, and an iso $\varphi : \xi_1|A \xrightarrow{\approx} \xi_2|A$, we constructed a v.b. $\xi_1 \cup_\varphi \xi_2 \to X$ by gluing together ξ_1 and ξ_2 by identifying $\xi_1|A$ and $\xi_2|A$ via φ.

We discussed

<u>Prop 1</u>: If $\varphi_0, \varphi_1 : \xi_1|A \xrightarrow{\approx} \xi_2|A$ are homotopic through isos $\xi_1|A \to \xi_2|A$, then $\xi_1 \cup_{\varphi_0} \xi_2 \approx \xi_1 \cup_{\varphi_1} \xi_2$.

and made the following remark

<u>Rk 2</u>: If $\xi_i = X_i \times \mathbb{F}^n$, $i=1,2$, then an iso $\xi_1|A \xrightarrow{\approx} \xi_2|A$ amounts to a map $A \to GL_n(\mathbb{F})$.
By Prop 1, the clutching construction for homotopic maps $A \to GL_n(\mathbb{F})$ gives isomorphic v.b.'s.

We call a map $A \to GL_n(\mathbb{F})$ a <u>clutching function</u>.

It is easy to check that the clutching construction has the following properties.

<u>Prop 3</u>:

1) $(\xi_1 \cup_\varphi \xi_2)|X_i \approx \xi_i$, $i=1,2$

2) If $\xi \to X$ is a v.b. and $\xi_i = \xi|X_i$, $i=1,2$, then $\xi_1 \cup_{id_{\xi|A}} \xi_2 \approx \xi$.

3) If $\beta_i : \xi_i \xrightarrow{\approx} \xi_i'$, $i=1,2$, are isos s.t. $\varphi'\beta_1 = \beta_2\varphi$, then $\xi_1 \cup_\varphi \xi_2 \approx \xi_1' \cup_{\varphi'} \xi_2'$.

4) Given clutching data (ξ_1, ξ_2, φ) and $(\xi_1', \xi_2', \varphi')$, we have

(i) $(\xi_1 \cup_\varphi \xi_2) \oplus (\xi_1' \cup_{\varphi'} \xi_2') \approx (\xi_1 \oplus \xi_1') \cup_{\varphi \oplus \varphi'} (\xi_2 \oplus \xi_2')$

(ii) $(\xi_1 \cup_\varphi \xi_2) \otimes (\xi_1' \cup_{\varphi'} \xi_2') \approx (\xi_1 \otimes \xi_1') \cup_{\varphi \otimes \varphi'} (\xi_2 \otimes \xi_2')$

(iii) $(\xi_1 \cup_\varphi \xi_2)^* \approx \xi_1^* \cup_{(\varphi^*)^{-1}} \xi_2^*$.

<u>Rk 4</u> In the case of trivial v.b.'s, Prop. 3.(4) translates as follows. Write $\varepsilon_i^k = X_i \times \mathbb{F}^k$ for the k-dim'l trivial v.b. over X_i, $i=1,2$. Let

$$\varphi : A \to GL_m(\mathbb{F}),$$
$$\psi : A \to GL_n(\mathbb{F}).$$

Then:

(i) $(\varepsilon_1^m \cup_\varphi \varepsilon_2^m) \oplus (\varepsilon_1^n \cup_\psi \varepsilon_2^n) \approx \varepsilon_1^{m+n} \cup_{\varphi \oplus \psi} \varepsilon_2^{m+n}$,

where
$$\varphi \oplus \psi : A \to GL_{m+n}(\mathbb{F})$$
$$a \mapsto \left(\begin{array}{c|c} \varphi(a) & 0 \\ \hline 0 & \psi(a) \end{array}\right).$$

(ii) Pick an identification $\mathbb{F}^{mn} = \mathbb{F}^m \otimes \mathbb{F}^n$. Then

$$(\mathcal{E}_1^m \cup_\varphi \mathcal{E}_2^m) \otimes (\mathcal{E}_1^n \cup_\psi \mathcal{E}_2^n) \approx \mathcal{E}_1^{mn} \cup_{\varphi\otimes\psi} \mathcal{E}_2^{mn}$$

where $\varphi \otimes \psi : A \longrightarrow GL_{mn}(\mathbb{F})$ sends $a \in A$ to the matrix of the map

$$\mathbb{F}^{mn} = \mathbb{F}^m \otimes \mathbb{F}^n \xrightarrow{\varphi(a)\otimes\psi(a)} \mathbb{F}^m \otimes \mathbb{F}^n = \mathbb{F}^{mn}.$$

(iii) $(\mathcal{E}_1^m \cup_\varphi \mathcal{E}_2^m)^* \approx \mathcal{E}_1^m \cup_{(\varphi^T)^{-1}} \mathcal{E}_2^m$

where

$$\begin{aligned} (\varphi^T)^{-1} : A &\longrightarrow GL_m(\mathbb{F}) \\ a &\longmapsto (\varphi(a)^T)^{-1} \end{aligned}$$

Exc: Check this!

The clutching construction is especially well suited for understanding v.b.'s over suspensions.

Def 5: The (unreduced) cone on a space X is the quotient space

$$CX = X \times I / X \times \{1\}.$$

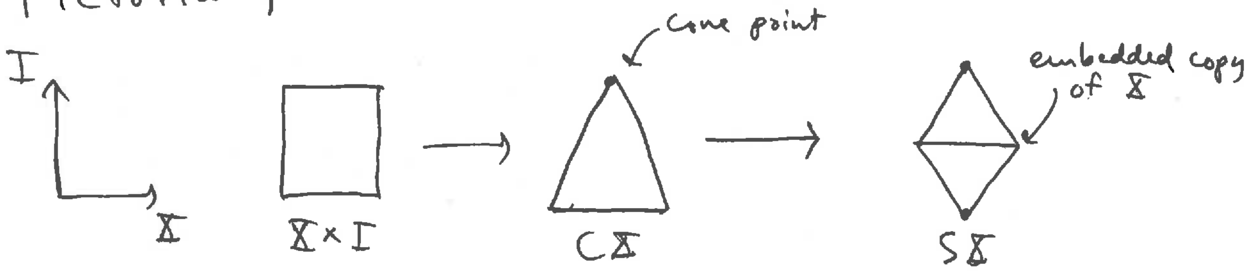
Identifying X as a subspace of CX via

$$\begin{aligned} X &\longrightarrow CX \\ x &\longmapsto [x,0] \end{aligned} \quad)$$

we define the (unreduced) suspension of X to be the quotient

$$SX = CX / X.$$

Pictorially:



We sometimes identify X with the "equator" of SX via the embedding

$$\begin{aligned} X &\longrightarrow SX \\ x &\longmapsto [x, \tfrac{1}{2}] \,. \end{aligned}$$

<u>Rk 6</u>: For any space X, the cone CX is contractible: the map

$$\begin{aligned} CX \times I &\longrightarrow CX \\ ([x,s], t) &\longmapsto [x, (1-t)s + t] \end{aligned}$$

is a deformation retraction onto the cone point.

<u>Rk 7</u>: We have $SX = C_+X \cup C_-X$, where

$$C_+X = (\text{image of } X \times [\tfrac{1}{2}, 1] \text{ in } SX)$$
$$C_-X = (\text{— " —} \; X \times [0, \tfrac{1}{2}] \; \text{— " —})$$

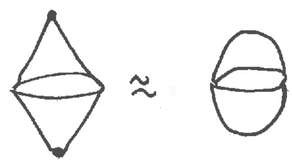
are both homeomorphic to CX. Notice that

$$C_+X \cap C_-X = X = X \times \{\tfrac{1}{2}\} \subset SX \,.$$

<u>Eg.</u> $CS^{n-1} \approx D^n \qquad [x,t] \longmapsto (1-t)x$

$SS^n \approx S^{n+1} \qquad C_+S^n \hookleftarrow$ upper hemisphere in S^{n+1}

$\qquad\qquad\qquad\quad C_-S^n \hookleftarrow$ lower — " — .



<u>Thm 8</u>: Sp. X is compact Hausdorff. Then the clutching construction gives a bijection

$$\Phi : [X, GL_n\mathbb{F}] \big/ _{\pi_0 GL_n\mathbb{F} \times \pi_0 GL_n\mathbb{F}} \xrightarrow{\approx} \mathrm{Vect}^n_{\mathbb{F}}(SX)$$

$$[\varphi] \longmapsto [\varepsilon^n_+ \cup_\varphi \varepsilon^n_-] .$$

Here

- $\varepsilon^n_\pm = C_\pm \times \mathbb{F}^n$ is the trivial n-dim'l v.b. $/ C_\pm X$
- For a space Z, we write $\pi_0 Z$ for the set of path components of Z; equivalently, $\pi_0 Z = [pt, Z]$.
- $\pi_0 GL_n\mathbb{F}$ has the group structure induced by the group structure on $GL_n\mathbb{F}$:
  $$[a][b] = [ab]$$
  for $a, b \in GL_n\mathbb{F}$.
- $\pi_0 GL_n\mathbb{F} \times \pi_0 GL_n\mathbb{F}$ acts on $[X, GL_n\mathbb{F}]$ by
  $$([a],[b]) \cdot [\varphi] = [a \cdot \varphi \cdot b^{-1}],$$
  where $a \cdot \varphi \cdot b^{-1} : X \to GL_n\mathbb{F}$, $x \mapsto a\varphi(x)b^{-1}$.
- The domain of Φ is the set of orbits for this action.

(<u>Note</u>: We restrict to compact Hausdorff spaces since that is the setting in which we have discussed the clutching construction. The theorem is true in much greater generality.)

<u>Pf of Thm 8:</u> By Prop. 1 and Prop. 3.(3) the map Φ is well-defined. Construct a map

$$\Psi : \mathrm{Vect}^n_{\mathbb{F}}(SX) \longrightarrow [X, GL_n\mathbb{F}]/\pi_0 GL_n(\mathbb{F})^2$$

as follows. Sp. $[\xi] \in \mathrm{Vect}^n_{\mathbb{F}}(SX)$. Since $C_\pm X$ is contractible, we can find a trivialization

$$h_\pm : \xi | C_\pm X \xrightarrow{\approx} \varepsilon^n_\pm .$$

Then the composite

$$\varepsilon^n_+ | X \xrightarrow{h_+^{-1}|} \xi | X \xrightarrow{h_-|} \varepsilon^n_- | X$$

is of the form

$$(x, v) \longmapsto (x, \varphi(h_+, h_-)(x) v)$$

for some

$$\varphi(h_+, h_-) : X \longrightarrow GL_n(\mathbb{F}).$$

Define $\Psi([\xi]) = [\varphi(h_+, h_-)] \in [X, GL_n\mathbb{F}]/\pi_0 GL_n(\mathbb{F})^2$

Let us verify that $\Psi([\xi])$ is independent of the choice of $h_\pm$. Sp. $h'_\pm$ are different choices. Then $h'_\pm = d_\pm \circ h_\pm$ for some automorphisms $d_\pm : \varepsilon^n_\pm \xrightarrow{\approx} \varepsilon^n_\pm$. Let $\delta_\pm : C_\pm X \to GL_n\mathbb{F}$ be the map corresponding to $d_\pm$. Then

$$\varphi(h'_+, h'_-)(x) = \delta_-(x)\, \varphi(h_+, h_-)(x)\, \delta_+(x)^{-1}$$

for all $x \in X$. Since $C_\pm X$ is contractible, $\delta_\pm : C_\pm X \to GL_n\mathbb{F}$ is homotopic to a constant map, say onto $a_\pm \in GL_n\mathbb{F}$. Thus $\varphi(h'_+, h'_-)$

is homotopic to the map

$$\begin{aligned} X &\longrightarrow GL_n \mathbb{F} \\ x &\longmapsto a_- \varphi(h_+, h_-)(x)\, a_+^{-1} \end{aligned} ,$$

whence

$$[\varphi(h_+', h_-')] = [\varphi(h_+, h_-)] \in [X, GL_n \mathbb{F}] / \pi_0 GL_n(\mathbb{F})^2 .$$

Finally, if ξ' is a v.b. isomorphic to ξ, pick an iso $\alpha : \xi' \xrightarrow{\approx} \xi$. Then $h_\pm \circ \alpha$ are trivializations for $\xi' | C_\pm X$ s.t.

$$\varphi(h_+ \circ \alpha, h_- \circ \alpha) = \varphi(h_+, h_-) .$$

Thus $\Psi([\xi])$ is independent of the choice of representative for the isomorphism class $[\xi]$.

Clearly

$$\Psi \Phi([\varphi]) = \Psi([\varepsilon_+^n \cup_\varphi \varepsilon_-^n]) = [\varphi]$$

for all φ. By construction, $[\varphi] = \Psi([\xi])$ is s.t. $\xi \approx \varepsilon_+^n \cup_\varphi \varepsilon_-^n$, so $\Phi\Psi([\xi]) = [\xi]$ for all ξ. Thus Φ and Ψ are inverse bijections. □

<u>Rk 9</u>: For all $n \geq 1$, the group $GL_n \mathbb{C}$ is path connected, while $GL_n \mathbb{R}$ has two path components, distinguished by the sign of the determinant.

(Sketch of pf: Notice that the following elementary row operations can be realized by a path in $GL_n \mathbb{F}$ in the sense that if B is the result of performing the operation on $A \in GL_n \mathbb{F}$, then A and B are connected by a path in $GL_n \mathbb{F}$:

1) adding a multiple of a row to another row
2) multiplying a row by a positive constant ($\mathbb{F} = \mathbb{R}$) or any non-zero constant ($\mathbb{F} = \mathbb{C}$).

Three operations of the first type allow one to realize the following operation:

3) swap two rows and multiply one of them by -1.

Two operations of type 3 allow one to realize

4) multiply two rows by -1.

Operations 1) - 4) allow one to construct a path from any $A \in GL_n \mathbb{F}$ to one of $\begin{pmatrix} \pm 1 & 0 \\ 0 & I_{n-1} \end{pmatrix}$. Moreover, in the case $\mathbb{F} = \mathbb{C}$, an operation of type 2 connects $\begin{pmatrix} -1 & 0 \\ 0 & I_{n-1} \end{pmatrix}$ to I_n.)

In view of Rk 9, Thm 8 gives

<u>Cor 10</u>: Sp. X is compact Hausdorff. Then

$$\mathrm{Vect}^n_{\mathbb{C}}(SX) \approx [X, GL_n \mathbb{C}]$$

$$\mathrm{Vect}^n_{\mathbb{R}}(SX) \approx [X, GL_n \mathbb{R}] / \mathbb{Z}/2 \times \mathbb{Z}/2 .$$

■

Eg: $\mathrm{Vect}^n_{\mathbb{C}}(S^1) \approx [S^0, GL_n\mathbb{C}] = pt$ for all n.
Thus all complex v.b.'s over S^1 are trivial.