Clutching construction (cont.)

Last time, we proved

Cor 1: Sp. X is a compact Hausdorff space. Then the clutching construction gives bijections

$$[X, GL_n\mathbb{C}] \longrightarrow Vect^n_{\mathbb{C}}(SX)$$

$$[X, GL_n\mathbb{R}]/\underbrace{\pi_0 GL_n(\mathbb{R})^2}_{\approx \mathbb{Z}/2\times\mathbb{Z}/2} \xrightarrow{\approx} Vect^n_{\mathbb{R}}(SX).$$

Rk 2: The Gram-Schmidt process gives deformation retractions from

$$\begin{cases} GL_n\mathbb{C} \text{ to } U(n) \subset GL_n\mathbb{C} \\ GL_n\mathbb{R} \text{ to } O(n) \subset GL_n\mathbb{R} \end{cases}.$$

So in the above we could replace $GL_n\mathbb{C}$ and $GL_n\mathbb{R}$ by $U(n)$ and $O(n)$, respectively. The advantage is that $U(n)$ and $O(n)$ are smaller spaces. For example,

- $U(1) = S^1 \subset \mathbb{C}$
- $O(1) = \{\pm 1\} \subset \mathbb{R}$
- $O(2)$ fits into a short exact sequence

$$1 \to SO(2) \to O(2) \xrightarrow{\det} \{\pm 1\} \to 1$$

and $SO(2) \approx S^1 \subset \mathbb{R}^2$.

Eg 3: Let us analyze the canonical line bundle $\gamma^1 \to \mathbb{C}P^1$ and its dual

$$H = (\gamma^1)^* = \mathrm{Hom}(\gamma^1, \varepsilon^1) \longrightarrow \mathbb{C}P^1.$$

Recall that $\mathbb{C}P^1 = \mathbb{C}^2 \setminus \{0\}/\sim$, where $\sim$ identifies (z_0, z_1) and $\lambda(z_0, z_1)$ for $\lambda \in \mathbb{C}\setminus\{0\}$. Write $[z_0 : z_1]$ ("homogeneous coordinates") for the equivalence class of (z_0, z_1).

The map

$$\begin{aligned} \mathbb{C}P^1 &\longrightarrow S^2 = \mathbb{C} \cup \{\infty\} \quad (\text{"Riemann sphere"}) \\ [z_0 : z_1] &\longmapsto z_0/z_1 \end{aligned}$$

is a homeomorphism, and we use it to identify $\mathbb{C}P^1$ and S^2. Let

$$D_0^2 = \{ z \in S^2 \mid |z| \leq 1 \}$$

$$D_\infty^2 = \{ z \in S^2 \mid |z| \geq 1 \}$$

(where $|\infty| = \infty$). Then D_0^2 and D_∞^2 are homeomorphic to D^2, $D_0^2 \cap D_\infty^2 = S^1$, and $D_0^2 \cup D_\infty^2 = S^2$. In homogeneous coordinates,

$$D_0^2 \ni z \longleftrightarrow [z : 1] \in \mathbb{C}P^1$$

$$D_\infty^2 \ni z \longleftrightarrow [1 : z^{-1}] \in \mathbb{C}P^1.$$

We have trivializations

$$\gamma^1 | D^2_0 \xrightarrow[\approx]{h_0} D^2_0 \times \mathbb{C} =: \varepsilon^1_0$$
$$(z, \lambda(z,1)) \longmapsto (z, \lambda)$$

$$\gamma^1 | D^2_\infty \xrightarrow[\approx]{h_\infty} D^2_\infty \times \mathbb{C} =: \varepsilon^1_\infty$$
$$(z, \lambda(1, z^{-1})) \longmapsto (z, \lambda).$$

Then

$$S^1 \times \mathbb{C} \xrightarrow[\approx]{h_0^{-1}|S^1} \gamma^1|S^1 \xrightarrow[\approx]{h_\infty|S^1} S^1 \times \mathbb{C}$$
$$(z, \lambda) \longmapsto (z, z\lambda)$$

So

$$\boxed{\gamma^1 \approx \varepsilon^1_0 \cup_\varphi \varepsilon^1_\infty \quad \text{for } \varphi : S^1 \to GL_1\mathbb{C},\ z \mapsto z.}$$

It follows by Rk X.4.(iii) that

$$\boxed{H \approx \varepsilon^1_0 \cup_\psi \varepsilon^1_\infty \quad \text{for } \psi : S^1 \to GL_1\mathbb{C},\ z \mapsto z^{-1}.}$$

Now by Rk X.4, the sum $H \oplus H$ has clutching function

$$\varphi_1 : S^1 \longrightarrow GL_2\mathbb{C},\quad z \longmapsto \begin{pmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{pmatrix},$$

while $H^{\otimes 2} \oplus \varepsilon^1$ has clutching function

$$\varphi_2 : S^1 \longrightarrow GL_2\mathbb{C},\quad z \longmapsto \begin{pmatrix} z^{-2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $GL_2\mathbb{C}$ is path connected, we can

find $\alpha : I \to GL_2\mathbb{C}$ s.t. $\alpha(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\alpha(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
Now the map

$$\begin{aligned} S^1 \times I &\longrightarrow GL_2\mathbb{C} \\ (z,t) &\longmapsto \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} \alpha(t) \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \alpha(t) \end{aligned}$$

is a homotopy from φ_1 to φ_2. Thus

$$\boxed{H \oplus H \approx H^{\otimes 2} \oplus \varepsilon^1 \quad \text{as v.b.'s over } \mathbb{C}P^1}$$

K-theory

So far, our vector bundles have been of constant dimension over the whole base space. We would now like to generalize slightly to allow vector bundles where the fibres may have different dimensions. From now we adopt the following definition

Def 4: An $\mathbb{F}$-vector bundle is a map $p : \xi \to B$ together with an $\mathbb{F}$-vector space structure on each fibre $\xi_b = p^{-1}(b)$, $b \in B$, s.t. the following local triviality condition is satisfied: for each $b_0 \in B$, there exists a neighbourhood U of b_0 s.t. there exists an $n \in \mathbb{Z}_{\geq 0}$ and a homeomorphism

$$h : \xi|U = p^{-1}(U) \longrightarrow U \times \mathbb{F}^n$$

which, for each $b \in U$, restricts to a linear isomorphism $\xi_b \xrightarrow{\approx} \{b\} \times \mathbb{F}^n$.

Notice that the only change w.r.t. the previous definition is that we no longer insist on n being the same for all local trivializations. By an n-dimensional v.b. we continue to mean a v.b. $\xi \to B$ with $\dim(\xi_b) = n$ for all $b \in B$, and these (for varying values of n) are precisely the v.b.'s in the previous sense.

Sp. $\xi \to B$ is a v.b. (in the sense of Def. 4).
Then the function

$$\begin{aligned} B &\longrightarrow \mathbb{Z}_{\geq 0} \\ b &\longmapsto \dim(\xi_b) \end{aligned}$$

is locally constant and hence continuous. Thus the subset

$$B_n = \{ b \in B \mid \dim(\xi_b) = n \} \subset B$$

is both open and closed for all n. We have $B = \coprod_{n \geq 0} B_n$ (as a space), and $\xi|B_n$ is an n-dim'l v.b., and hence a v.b. in the previous sense.

All our work so far generalizes to v.b.'s in the sense of Def 4.

<u>K-theory as an abelian group</u>

Write $\mathrm{Vect}_{\mathbb{F}}(X)$ for the set of isomorphism classes of numerable $\mathbb{F}$-v.b.'s over X.

Notice that $\oplus$ makes $\mathrm{Vect}_{\mathbb{F}}(X)$ into a commutative monoid (i.e. a set equipped with a binary operation which is associative, commutative and has a neutral element). $\mathrm{Vect}_{\mathbb{F}}(X)$ fails to be an abelian group, however, since vector bundles do not have inverses w.r.t. $\oplus$.

Abelian groups are easier to understand than commutative monoids, so we would like to turn $\mathrm{Vect}_{\mathbb{F}}(X)$ into an abelian group by formally adjoining additive inverses.

Def 5: Sp. $M = (M, +)$ is a commutative monoid. The Grothendieck group of M is the abelian group $Gr(M)$ constructed as follows:

- as a set, $Gr(M) = M \times M / \sim$, where $(x,y) \sim (x',y') \Leftrightarrow \exists z \in M$ s.t. $x + y' + z = x' + y + z$
- the group structure on $Gr(M)$ is given by $[(x,y)] + [(x',y')] = [(x+x', y+y')]$.

(One should think of $[(x,y)]$ as $x - y$.)

Exc: Check that addition on $Gr(M)$ is well defined and makes $Gr(M)$ into an abelian group with identity element $[(0,0)]$ and $-[(x,y)] = [(y,x)]$.

Notice that the construction of $Gr(M)$ generalizes the usual construction of integers from natural numbers, so we have $Gr(\mathbb{N}, +) = (\mathbb{Z}, +)$. (Check this if you haven't seen it!)

We have a homomorphism of commutative monoids

$$\begin{aligned} i: M &\longrightarrow Gr(M) \\ x &\longmapsto [(x,0)] \end{aligned}$$

(not necessarily injective). This has the following <u>universal property</u>:

<u>Prop 6</u>: Sp. A is an abelian group, and let $f: M \to A$ be a homomorphism of commutative monoids. Then there exists a unique homomorphism $F: Gr(M) \to A$ of abelian groups s.t. $f = F \circ i$.

$$\begin{tikzcd} M \arrow[r, "i"] \arrow[dr, "f"'] & Gr(M) \arrow[d, dashed, "\exists!\, F"] \\ & A \end{tikzcd}$$

<u>Sketch of pf</u>: We have $[(x,y)] = i(x) - i(y)$, so we must define

$$F([(x,y)]) = f(x) - f(y).$$

Now check that this def'n does give a well-defined homomorphism. $\square$

<u>Interpretation</u>: The homomorphism $i: M \to Gr(M)$ is the "best" homomorphism from M to an abelian group.

<u>Rk 7</u>: The universal property characterizes $Gr(M)$ uniquely up to unique isomorphism: if $j: M \to G$ is another monoid homomorphism into an abelian

group satisfying the universal property, then the maps $I : G \to Gr(M)$ and $J : Gr(M) \to G$ induced by i and j are inverse isomorphisms. (Check this!)

Def 8: For X a compact Hausdorff space, we define the $K_{\mathbb{F}}$-theory of X to be

$$K_{\mathbb{F}}(X) = Gr(Vect_{\mathbb{F}}(X), \oplus).$$

Notation: One usually writes K or KU for $K_{\mathbb{C}}$ (complex K-theory) and KO for $K_{\mathbb{R}}$ (real K-theory).

We write $[\xi]$ (or even just ξ) for the element of $K_{\mathbb{F}}(X)$ defined by a v.b. $\xi \to X$. Then in $K_{\mathbb{F}}(X)$, we have $[\xi] + [\zeta] = [\xi \oplus \zeta]$ for v.b.'s $\xi, \zeta \to X$.

Eg: $(Vect_{\mathbb{F}}(pt), \oplus) \approx (\mathbb{N}, +)$, so $K_{\mathbb{F}}(pt) \approx \mathbb{Z}$.