Recap of last time

For M a commutative monoid, we constructed an abelian group $Gr(M)$, the Grothendieck group of M. As a set,

$$Gr(M) = M \times M / \sim ,$$

where $(x,y) \sim (x',y') \Leftrightarrow \exists z \in M$ s.t. $x + y' + z = x' + y + z$.
Intuitively, one should think of $[(x,y)]$ as a formal difference $x - y$.

For X a compact Hausdorff space, we defined

$$K_{\mathbb{F}}(X) = Gr(\mathrm{Vect}_{\mathbb{F}}(X), \oplus).$$

One usually writes K or KU for $K_{\mathbb{C}}$ and KO for $K_{\mathbb{R}}$. We write $[\xi]$ for $[([\xi],0)] \in K_{\mathbb{F}}(X)$ when ξ is a v.b. over X.

Remarks

Rk 1: $[([\xi],[\zeta])] = [\xi] - [\zeta] \in K_{\mathbb{F}}(X)$, so every element of $K_{\mathbb{F}}(X)$ can be written as a difference of v.b.'s. Since X is compact Hausdorff, every v.b. over X admits a complement. Thus every element of $K_{\mathbb{F}}(X)$ can be written as $[\xi] - [\varepsilon^n]$ for some v.b. $\xi \to X$ and $n \geq 0$.

<u>Def 2:</u> Call v.b.'s $\xi, \zeta \to X$ <u>stably</u> <u>isomorphic</u>, $\xi \approx_s \zeta$, if $\exists n \geq 0$ s.t. $\xi \oplus \varepsilon^n \approx \zeta \oplus \varepsilon^n$.

<u>Rk 3:</u> We have

$$[\xi] - [\zeta] = [\xi'] - [\zeta'] \quad \text{in } K_{\mathbb{F}}(X)$$

$$\Leftrightarrow [([\xi],[\zeta])] = [([\xi'],[\zeta'])] \quad \text{— " —}$$

$$\Leftrightarrow [\xi] \oplus [\zeta'] \oplus [\eta] = [\xi'] \oplus [\zeta] \oplus [\eta] \quad \text{in } \mathrm{Vect}_{\mathbb{F}}(X) \text{ for some } \eta \to X$$

$$\Leftrightarrow [\xi] \oplus [\zeta'] \oplus [\varepsilon^n] = [\xi'] \oplus [\zeta] \oplus [\varepsilon^n] \quad \text{in } \mathrm{Vect}_{\mathbb{F}}(X) \text{ for some } n \geq 0$$

since $\eta \to X$ admits a complement

$$\Leftrightarrow \xi \oplus \zeta' \approx_s \xi' \oplus \zeta .$$

<u>Product structure</u>

In addition to $\oplus$, $\mathrm{Vect}_{\mathbb{F}}(X)$ has another binary operation, namely $\otimes$. Properties:

- $(\mathrm{Vect}_{\mathbb{F}}, \oplus)$ is a commutative monoid
- $(\mathrm{Vect}_{\mathbb{F}}, \otimes)$ — " —
- $\otimes$ distributes over $\oplus$
- $0 \otimes x = 0$ for all $x \in \mathrm{Vect}_{\mathbb{F}}$ (where $0 = [\varepsilon^0]$ is the neutral element for $\oplus$).

Such a structure is called a <u>commutative</u> <u>semiring</u>.

Prop 4: Sp. $(M, +, \cdot)$ is a commutative semiring. Then $Gr(M, +)$ is a commutative ring with multiplication

$$[(x,y)]\,[(x',y')] = [(xx'+yy', xy'+x'y)].$$

Pf: Exercise. □

The map $i: M \longrightarrow Gr(M)$, $x \mapsto [(x,0)]$ is a homomorphism of commutative semirings.

Cor 5: $K_{\mathbb{F}}(X) = Gr(\mathrm{Vect}_{\mathbb{F}}(X), \oplus)$ is a comm. ring with $[\xi][\varsigma] = [\xi \otimes \varsigma] \in K_{\mathbb{F}}(X)$ for v.b.'s $\xi, \varsigma \to X$. □

Induced maps

For a homomorphism $f: M \to N$ of commutative monoids, the universal property of $Gr(M)$ implies the existence of a unique homomorphism of abelian groups

$$Gr(f): Gr(M) \longrightarrow Gr(N)$$

s.t. the square

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "i"'] & N \arrow[d, "i"] \\ Gr(M) \arrow[r, dashed, "Gr(f)"] & Gr(N) \end{tikzcd}$$

commutes. Concretely, $Gr(f)([(x,y)]) = [(f(x), f(y))]$.

If M and N are commutative semirings, then $Gr(f)$ is a ring homomorphism. Notice that

$$\begin{cases} Gr(id_M) = id_{Gr(M)} \\ Gr(f \circ g) = Gr(f) \circ Gr(g) \end{cases}$$

(Gr is a <u>covariant functor</u>)

<u>Def 6</u>: Sp. $f: X \to Y$ is a continuous map between compact Hausdorff spaces. We define the <u>induced map</u>

$$f^*: K_{\mathbb{F}}(Y) \longrightarrow K_{\mathbb{F}}(X)$$

to be the ring homomorphism

$$f^* = Gr\left(f^*: Vect_{\mathbb{F}}(Y) \longrightarrow Vect_{\mathbb{F}}(X) \right)$$

(so $f^*[\xi] = [f^*\xi] \in K_{\mathbb{F}}(X)$ for ξ a v.b. over Y).

Notice that

$$\begin{cases} id_X^* = id_{K_{\mathbb{F}}(X)} \\ (f \circ g)^* = g^* \circ f^* \end{cases}$$

($K_{\mathbb{F}}$ is a <u>contravariant functor</u>)

<u>Prop 7</u>: If $f, g: X \to Y$ are homotopic, then $f^* = g^*: K_{\mathbb{F}}(Y) \longrightarrow K_{\mathbb{F}}(X)$.

<u>Pf</u>: This is already true on the level of $Vect_{\mathbb{F}}$. $\square$

Prop. 7 has the following formal corollary:

Cor 8: If $f: X \to Y$ is a homotopy equivalence, then $f^*: K_{\mathbb{F}}(Y) \to K_{\mathbb{F}}(X)$ is an isomorphism.

Pf: Let $g: Y \to X$ be a homotopy inverse for f, so that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. Now

$$f^* \circ g^* = (g \circ f)^* = id_X^* = id_{K_{\mathbb{F}}(X)}$$

and

$$g^* \circ f^* = (f \circ g)^* = id_Y^* = id_{K_{\mathbb{F}}(Y)},$$

so f^* and g^* are inverse isomorphisms. $\square$

K-theory of disjoint unions

Prop 9: Sp. X and Y are compact Hausdorff spaces. Then the inclusions $i: X \hookrightarrow X \amalg Y$ and $j: Y \to X \amalg Y$ induce a ring isomorphism

$$K_{\mathbb{F}}(X \amalg Y) \xrightarrow[\approx]{(i^*, j^*)} K_{\mathbb{F}}(X) \times K_{\mathbb{F}}(Y).$$

Pf: Clearly

$$Vect_{\mathbb{F}}(X \amalg Y) \xrightarrow{(i^*, j^*)} Vect_{\mathbb{F}}(X) \times Vect_{\mathbb{F}}(Y)$$

is an iso of comm. semirings. Moreover, for comm. semirings M_1 and M_2, the projections $\pi_i: M_1 \times M_2 \to M_i$, $i = 1, 2$, induce an iso

$$Gr(M_1 \times M_2) \xrightarrow[\approx]{(Gr(\pi_1), Gr(\pi_2))} Gr(M_1) \times Gr(M_2).$$

The claim follows. $\square$

Pairs of spaces and pointed spaces

Def 10: A *pair of spaces* is a pair (X, A) where X is a space and $A \subset X$ is a subspace. A *map* $f:(X,A) \longrightarrow (Y,B)$ between pairs of spaces is a continuous map $f: X \to Y$ s.t. $fA \subset B$. If $f,g:(X,A) \longrightarrow (Y,B)$, a *homotopy* h from f to g is a map

$$h:(X \times I, A \times I) \longrightarrow (Y, B)$$

s.t. $h(-,0) = f$ and $h(-,1) = g$. In other words, h is an ordinary homotopy from f to g s.t. $h(-,t)$ is a map of pairs $(X,A) \to (Y,B)$ for all $t \in I$. From the notion of homotopy between maps of pairs, mimicking the definitions for spaces (lecture 5), we obtain the notions of *homotopic* maps of pairs (written $f \simeq g$), *homotopy class* of maps of pairs (we write $[f]$ for the homotopy class of $f:(X,A) \to (Y,B)$), *homotopy equivalent* pairs (written $(X,A) \simeq (Y,B)$), etc. For example, pairs (X,A) and (Y,B) are homotopy equivalent if $\exists$ maps

$$f:(X,A) \longrightarrow (Y,B) \quad \text{and} \quad g:(Y,B) \longrightarrow (X,A)$$

s.t. $f \circ g \simeq id_{(Y,B)}$ and $g \circ f \simeq id_{(X,A)}$ as maps of pairs. We write $[(X,A),(Y,B)]$ for the set of homotopy classes of maps $(X,A) \to (Y,B)$.

Sometimes we identify a space X with the pair $(X, \emptyset)$.

Notation:

$$(X,A) \times (Y,B) := (X \times Y, X \times B \cup A \times Y)$$

$$(X,A) \times Y := (X,A) \times (Y,\emptyset) = (X \times Y, A \times Y)$$

$$X \times (Y,B) := (X,\emptyset) \times (Y,B) = (X \times Y, X \times B).$$

Pairs (X,A) where A consists of a single point are of particular interest.

Def 11: A pointed (or based) space is a pair (X, x_0) where X is a space and $x_0 \in X$ is a point of X, the basepoint. We identify (X, x_0) with the pair $(X, \{x_0\})$. Specializing Def. 10 to the case $A = \{x_0\}$, $B = \{y_0\}$, we obtain the notions of pointed (or based or basepoint-preserving) map between pointed spaces, homotopy between maps of pointed spaces, etc. Explicitly, a basepoint-preserving homotopy from $f: (X,x_0) \to (Y,y_0)$ to $g: (X,x_0) \to (Y,y_0)$ is an ordinary homotopy h from f to g with the property that

$$h(x_0, t) = y_0$$

for all $t \in I$. We write $f \simeq g$ to indicate that based maps f and g are based homotopic; $[f]$ for the based homotopy class of f; $(X,x_0) \simeq (Y,y_0)$

to indicate that (X, x_0) and (Y, y_0) are homotopy equivalent as pointed spaces; and $[(X, x_0); (Y, y_0)]$ for the set of pointed homotopy classes of maps $(X, x_0) \to (Y, y_0)$.

Often the basepoint is left implicit in the notation, and we write X for (X, x_0). This unfortunately makes the notation $[X, Y]$ ambiguous — one has to rely on context to determine whether it means pointed or ordinary (= <u>free</u>) homotopy classes of maps $X \to Y$.