Pairs and pointed spaces (cont.)

Def 1: If X is a space, we write X_+ for the pointed space $(X \amalg pt, pt)$ (so X_+ is the space X with a disjoint basepoint added).

For a pair (X, A), we can form the quotient space X/A, and take as a basepoint the point corresponding to A. In the case $A = \emptyset$, we set $X/\emptyset = X_+$

Obs 2: A map $(X, A) \to (Y, B)$ induces a pointed map $X/A \to Y/B$. The maps induced by homotopic maps of pairs are pointed homotopic.

Reduced and relative K-theory

Def 3: Sp. $X = (X, x_0)$ is a pointed compact Hausdorff space. We define the reduced $K_{\mathbb{F}}$-theory of X to be

$$\tilde{K}_{\mathbb{F}}(X) := \operatorname{Ker}\left(K_{\mathbb{F}}(X) \xrightarrow{i^*} K_{\mathbb{F}}(x_0)\right)$$

where i denotes the inclusion $i : \{x_0\} \hookrightarrow X$.

Equivalently,

$$\tilde{K}_{\mathbb{F}}(X) = \mathrm{Ker}\,(K_{\mathbb{F}}(X) \xrightarrow{d} \mathbb{Z})$$

where d is the map given by $d([\xi]) = \dim(\xi_{x_0})$ for v.b.'s ξ.

<u>Def 4</u>: Sp. X is a compact Hausdorff space and $A \subset X$ is a closed subspace (i.e. (X, A) is a <u>compact pair</u>). We define the <u>$K_{\mathbb{F}}$-theory of the pair</u> (X, A) (or the <u>$K_{\mathbb{F}}$-theory of X relative to A</u>) to be

$$K_{\mathbb{F}}(X, A) := \tilde{K}_{\mathbb{F}}(X/A).$$

<u>Rk 5</u>: Notice that $\tilde{K}_{\mathbb{F}}(X)$ inherits a multiplication from $K(X)$, but in general this product is nonunital ($\tilde{K}_{\mathbb{F}}(X)$ is a "ring without a unit").

We will now attempt to quantify the difference between $K_{\mathbb{F}}(X)$ and $\tilde{K}_{\mathbb{F}}(X)$. The following is a standard result concerning short exact sequences.

<u>Lemma 6</u>: Suppose

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0 \qquad (1)$$

is a short exact sequence of abelian groups (or R-modules, ...). Then the following are equivalent:

(i) $\exists$ homomorphism $s : C \to B$ s.t. $ps = \mathrm{id}_C$
(a <u>section</u> for p)

(ii) $\exists$ homomorphism $r : B \to A$ s.t. $ri = \mathrm{id}_A$
(a <u>retraction</u> for i)

(iii) $\exists$ isomorphism $\theta : B \longrightarrow A \oplus C$ s.t. the diagram

$$\begin{tikzcd}
& & B \arrow[dr, "p"] \arrow[dd, "\approx"', "\theta"] & & \\
0 \arrow[r] & A \arrow[ur, "i"] \arrow[dr, "i_A"'] & & C \arrow[r] & 0 \\
& & A \oplus C \arrow[ur, "p_C"'] & &
\end{tikzcd}$$

commutes, where $i_A : a \mapsto (a,0)$ and $p_C : (a,c) \mapsto c$ are the evident inclusion and projection maps.

<u>Pf</u>: (iii) $\Rightarrow$ (i) & (ii): Given θ, we can define $s = \theta^{-1} i_C$ and $r = p_A \theta$ where $i_C : C \to A \oplus C$, $c \mapsto (0,c)$ and $p_A : A \oplus C \longrightarrow A$, $(a,c) \mapsto a$ are the evident maps.

(i) $\Rightarrow$ (ii): Given s, define a map $\tilde{r} : B \to B$ by setting $\tilde{r}(b) = b - sp(b)$. Then for all $b \in B$,

$$p\tilde{r}(b) = p(b - sp(b)) = p(b) - psp(b) = p(b) - p(b) = 0,$$

so $\tilde{r}$ lifts to a map $r : B \to A$ s.t. $ir = \tilde{r}$.

Now

$$iri(a) = \tilde{r}i(a) = i(a) - spi(a) = i(a)$$

for all $a \in A$. Since i is a monomorphism, it follows that $ri = \mathrm{id}_A$.

(ii) $\Rightarrow$ (i): Given r, define a map $\tilde{s}: B \to B$ by setting $\tilde{s}(b) = b - ir(b)$. Then for all $a \in A$,

$$\tilde{s}i(a) = i(a) - iri(a) = i(a) - i(a) = 0,$$

so $\tilde{s}$ descends to a map $s: C \to B$ s.t.: $\tilde{s} = sp$. Now

$$psp(b) = p\tilde{s}(b) = p(b) - pir(b) = p(b)$$

for all $b \in B$. Since p is an epimorphism, it follows that $ps = id_C$.

(i) & (ii) $\Rightarrow$ (iii): Define $\theta: B \longrightarrow A \oplus C$ by $\theta(b) = (r(b), p(b))$. Then θ makes the diagram in (iii) commutative. If $\theta(b) = 0$, then $p(b) = 0$ and $r(b) = 0$. From the former we get that $b = i(a)$ for some $a \in A$, and then the latter gives $a = ri(a) = 0$. Thus $b = i(a) = 0$, and θ is a monomorphism. To see that θ is an epimorphism, sp. $(a,c) \in A \oplus C$. Now

$$\begin{aligned}
&\theta(i(a - rs(c)) + s(c)) \\
&= (ri(a - rs(c)) + rs(c),\ pi(a - rs(c)) + ps(c)) \\
&= (a - rs(c) + rs(c),\ ps(c)) \\
&= (a, c).
\end{aligned}$$

The claim follows. $\square$

Def 7: If $\exists$ s or r as in Lemma 6, the short exact sequence (1) is called split.

Now let (X, x_0) be a pointed compact Hausdorff space. The maps induced by the inclusion $i : \{x_0\} \hookrightarrow X$ and the unique map $r : X \to \{x_0\}$ satisfy $i^* r^* = id_{K_{\mathbb{F}}(x_0)}$, so we get a split short exact sequence

$$0 \to \tilde{K}_{\mathbb{F}}(X) \longrightarrow K_{\mathbb{F}}(X) \underset{r^*}{\overset{i^*}{\rightleftarrows}} K_{\mathbb{F}}(x_0) \longrightarrow 0.$$

Since $K_{\mathbb{F}}(x_0) \approx \mathbb{Z}$, we get an isomorphism of ab. groups

$$\boxed{K_{\mathbb{F}}(X) \approx \tilde{K}_{\mathbb{F}}(X) \oplus \mathbb{Z}}$$

We now turn to the construction of induced maps. If $f : (X, x_0) \to (Y, y_0)$ is a map of compact pointed Hausdorff spaces, the map

$$f^* : K_{\mathbb{F}}(Y) \longrightarrow K_{\mathbb{F}}(X)$$

restricts to give an induced map

$$f^* : \tilde{K}_{\mathbb{F}}(Y) \longrightarrow \tilde{K}_{\mathbb{F}}(X). \qquad (2)$$

Diagrammatically, we have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \tilde{K}_{\mathbb{F}}(Y) \arrow[r] \arrow[d, dashed, "f^*"'] & K_{\mathbb{F}}(Y) \arrow[r, "i^*"] \arrow[d, "f^*"] & K_{\mathbb{F}}(y_0) \arrow[d, "(f|)^*"] \\ 0 \arrow[r] & \tilde{K}_{\mathbb{F}}(X) \arrow[r] & K_{\mathbb{F}}(X) \arrow[r] & K_{\mathbb{F}}(x_0) \end{tikzcd}$$

The map (2) only depends on the based homotopy class of f.

If $f : (X, A) \longrightarrow (Y, B)$, we define the induced map

$$f^* : K_{\mathbb{F}}(Y, B) \longrightarrow K_{\mathbb{F}}(X, A) \qquad (3)$$

to be the map

$$f^* = \hat{f}^* : \tilde{K}_{\mathbb{F}}(Y/B) \longrightarrow \tilde{K}_{\mathbb{F}}(X/A)$$

where $\hat{f} : X/A \longrightarrow Y/B$ is the map defined by f. The map (3) only depends on the homotopy class of f as a map $(X, A) \to (Y, B)$.

<u>Thm 8</u> (Excision): Sp. (X, A) is a compact pair, and let $U \subset X$ be an open subset s.t. $\bar{U} \subset A$. Then the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism

$$K_{\mathbb{F}}(X, A) \xrightarrow{\approx} K_{\mathbb{F}}(X \setminus U, A \setminus U).$$

<u>Pf</u>: $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces a homeomorphism

$$(X \setminus U)/(A \setminus U) \xrightarrow{\approx} X/A. \qquad \square$$

Rk 9: From the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \tilde{K}_{\mathbb{F}}(X_+) \arrow[r] \arrow[d, dashed] & K_{\mathbb{F}}(X_+) \arrow[r, "i^*"] \arrow[d, "\approx"', "\text{Prop. XII.9}"] & K_{\mathbb{F}}(pt) \arrow[d, equal] \\ 0 \arrow[r] & K_{\mathbb{F}}(X) \arrow[r] & K_{\mathbb{F}}(X) \times K_{\mathbb{F}}(pt) \arrow[r, "pr"] & K_{\mathbb{F}}(pt) \end{tikzcd}$$

($i : pt \hookrightarrow X_+$ the inclusion of the basepoint) we get a natural isomorphism

$$\boxed{\tilde{K}_{\mathbb{F}}(X_+) \xrightarrow{\approx} K_{\mathbb{F}}(X).}$$

From this isomorphism, we get a natural iso

$$\boxed{K_{\mathbb{F}}(X, \emptyset) = \tilde{K}_{\mathbb{F}}(X/\emptyset) = \tilde{K}_{\mathbb{F}}(X_+) \approx K_{\mathbb{F}}(X).}$$

(This is compatible with the idea that we should be able to treat a space X as a pair $(X, \emptyset)$.)

The reduced K-groups have a nice interpretation in terms of stable equivalence classes of bundles.

Def 10: Call v.b.'s $\xi, \zeta \to X$ stably equivalent, $\xi \sim_s \zeta$, if $\exists\, m, n \geq 0$ s.t. $\xi \oplus \varepsilon^m \approx \zeta \oplus \varepsilon^n$.

It is easy to verify that $\sim_s$ is an equivalence relation and that $\sim_s$ defines an equivalence relation on $\mathrm{Vect}_{\mathbb{F}}(X)$. Write $[\xi]_s$ for the $\sim_s$-equivalence class of ξ.

Notice that Whitney sum gives a well-defined addition on $\mathrm{Vect}_{\mathbb{F}}(X)/\sim_s$ with neutral el't $0 = [\varepsilon^0]_s$. If X is compact Hausdorff, then any v.b. $\xi \to X$ admits a complement ς with $\xi \oplus \varsigma \approx \varepsilon^n$ for some $n \geq 0$, and then

$$[\xi]_s + [\varsigma]_s = [\xi \oplus \varsigma]_s = [\varepsilon^n]_s = [\varepsilon^0]_s = 0$$

in $\mathrm{Vect}_{\mathbb{F}}(X)/\sim_s$, so that $[\varsigma]_s = -[\xi]_s$. Thus, in this case, $\mathrm{Vect}_{\mathbb{F}}(X)/\sim_s$ is an abelian group.

<u>Prop 11</u>: So. (X, x_0) is a pointed compact Hausdorff space. Then the map

$$\varphi : \mathrm{Vect}_{\mathbb{F}}(X)/\sim_s \longrightarrow \tilde{K}_{\mathbb{F}}(X)$$
$$[\xi]_s \longmapsto [\xi] - [\varepsilon^{\dim(\xi_{x_0})}]$$

is an isomorphism of abelian groups.

<u>Sketch of pf</u>: It is easy to check that φ is a well-defined homomorphism. It follows from Rk XII.1 that φ is an epi, and from Rk XII.3 that φ is a mono. $\square$

Constructions on pointed spaces

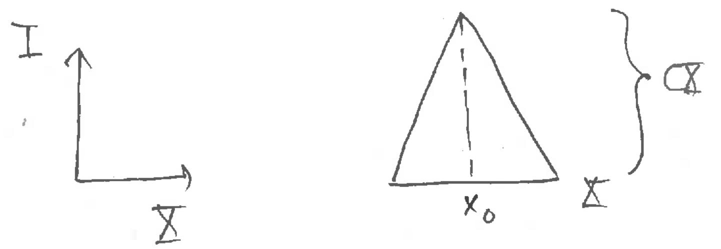
Def 12: Let $X = (X, x_0)$ be a pointed space. We define the (reduced) cone on X to be the quotient

$$CX = C(X, x_0) = \frac{X \times I}{X \times \{1\} \cup \{x_0\} \times I}$$

equipped with the basepoint given by the collapsed subspace. (Thus the reduced cone is obtained from the unreduced cone by collapsing the line $\{x_0\} \times I$ to a point.) The map

$$\begin{aligned} X &\longrightarrow CX \\ x &\longmapsto [x, 0] \end{aligned}$$

is a pointed embedding, and we use it to identify X with a subspace of CX. Picture:



(The dashed line is collapsed to a point.)

We use the same notation for the reduced and unreduced cones, and rely on context to make it clear which one is intended.

Lemma 13: For any pointed space (X, x_0), the cone $C(X, x_0)$ deformation retracts onto its basepoint.

Pf: The map

$$\begin{aligned} C(X, x_0) \times I &\longrightarrow C(X, x_0) \\ ([x, s], t) &\longmapsto [x, (1-t)s + t] \end{aligned}$$

provides the deformation retraction. $\square$

Thus all reduced cones are pointed homotopy equivalent to the one-point space.