Constructions on based spaces (cont.)

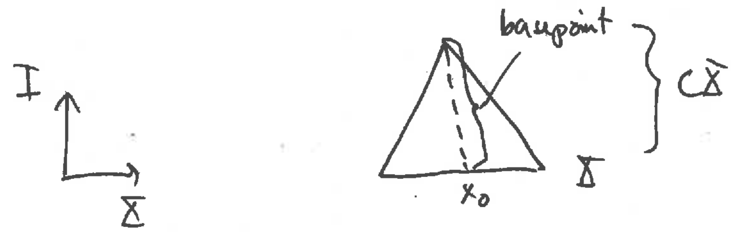
<u>Recall</u>: Last time, we defined the <u>reduced cone</u> of a pointed space $X = (X, x_0)$ as the quotient

$$CX = C(X, x_0) = \frac{X \times I}{X \times \{1\} \cup \{x_0\} \times I}$$

equipped with the basepoint given by the collapsed subspace. We identify X with the base of the cone via the embedding

$$\begin{aligned} X &\longrightarrow CX \\ x &\longmapsto [x, 0] \end{aligned}$$

Picture:



We saw that CX is always pointed contractible.

<u>Rk 1</u>: The reduced cone $C(X_+)$ is homeomorphic to the unreduced cone CX.

<u>Exc 2</u>: Pick a basepoint $x_0 \in S^{n-1}$. Show that $C(S^{n-1}, x_0)$ is homeomorphic to the closed n-disk D^n.

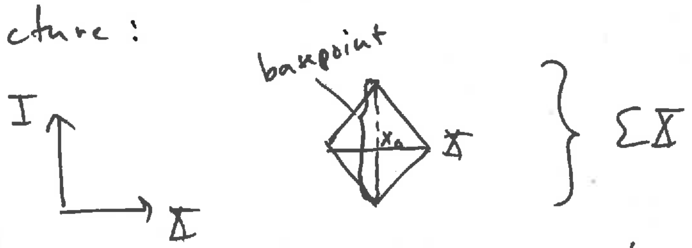
Def 3: The *reduced suspension* of a pointed space $X = (X, x_0)$ is the quotient

$$\Sigma X = \Sigma(X, x_0) = C(X, x_0)/X$$

equipped with the basepoint given by the collapsed subspace. (Alternatively, $\Sigma X = SX/\{x_0\}\times I$.) Notice that X embeds into ΣX as the "equator" via the map

$$\begin{aligned} X &\longrightarrow \Sigma X \\ x &\longmapsto [x, 1/2] \end{aligned}$$

Picture:



We can view ΣX as the union of two reduced cones intersecting along the equator.

Rk 4: $\Sigma(X_+) \not\cong SX$ in general.

Exc 5: Show that ΣS^{n-1} is homeomorphic to S^n for all $n \geq 1$.

Def 6: If $f : (X, x_0) \to (Y, y_0)$, we define pointed maps

$$Cf : C(X, x_0) \longrightarrow C(Y, y_0), \quad [x, t] \mapsto [f(x), t]$$
$$\Sigma f : \Sigma(X, x_0) \longrightarrow \Sigma(Y, y_0), \quad [x, t] \mapsto [f(x), t].$$

Rk 8: The wedge sum $(X,x_0)\vee(Y,y_0)$ is the categorical coproduct of (X,x_0) and (Y,y_0) in the category of pointed spaces and pointed maps: the inclusions

$$(X,x_0) \xhookrightarrow{i_X} (X,x_0)\vee(Y,y_0)$$
$$(Y,y_0) \xhookrightarrow{i_Y} (X,x_0)\vee(Y,y_0)$$

have the property that for any maps

$$(X,x_0) \xrightarrow{f} (Z,z_0)$$
$$(Y,y_0) \xrightarrow{g} (Z,z_0),$$

there exists a unique map

$$(X,x_0)\vee(Y,y_0) \xrightarrow{(f,g)} (Z,z_0)$$

s.t. $(f,g)\circ i_X = f$ and $(f,g)\circ i_Y = g$.

Notice that $(X,x_0)\vee(Y,y_0)$ embeds into $X\times Y$ as the subspace

$$\{(x,y)\in X\times Y \mid x=x_0 \text{ or } y=y_0\} \subset X\times Y.$$

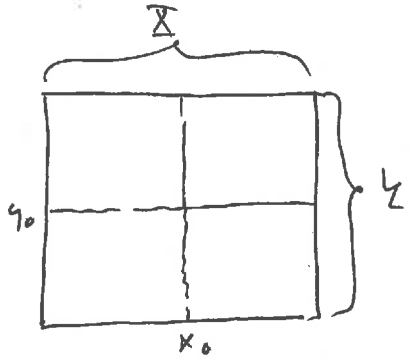
Def 9: The *smash product* of $X=(X,x_0)$ and $Y=(Y,y_0)$ is the quotient

$$X\wedge Y = (X,x_0)\wedge(Y,y_0) = X\times Y / X\vee Y$$

equipped with the basepoint given by the collapsed subspace. For $(x,y)\in X\times Y$, we write $x\wedge y$ for $[(x,y)]\in X\wedge Y$.

Picture:

$X \wedge Y =$



(= basepoint)

Eg. 10: (i) $C(X, x_0) = (X, x_0) \wedge (I, 1)$

(ii) $\Sigma(X, x_0) \approx (X, x_0) \wedge (S^1, 1)$

(Reason: viewing S^1 as $I/\partial I$, both sides amount to the quotient

$$X \times I \big/ X \times \{0,1\} \cup \{x_0\} \times I \, . \quad)$$

(iii) $X_+ \wedge Y_+ \approx (X \times Y)_+$

(iv) $(X, x_0) \wedge (S^0, 1) \approx (X, x_0)$

(v) a pointed homotopy

$$X \times I \longrightarrow Y$$

is the same data as a pointed map

$$X \wedge I_+ \longrightarrow Y \, .$$

Lemma 11: If X, Y, Z are compact Hausdorff spaces. Then the map

$$a : (X \wedge Y) \wedge Z \longrightarrow X \wedge (Y \wedge Z)$$
$$(x \wedge y) \wedge z \longmapsto x \wedge (y \wedge z)$$

is a homeomorphism.

Pf: The map a fits into the commutative triangle

$$\begin{tikzcd} & X\times Y\times Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ (X\wedge Y)\wedge Z \arrow[rr, "a"] & & X\wedge(Y\wedge Z) \end{tikzcd}$$

where π_1 is the composite

$$X\times Y\times Z \xrightarrow{\text{quot.}\times 1} (X\wedge Y)\times Z \xrightarrow{\text{quot.}} (X\wedge Y)\wedge Z$$

and π_2 the composite

$$X\times Y\times Z \xrightarrow{1\times\text{quot.}} X\times (Y\wedge Z) \xrightarrow{\text{quot.}} X\wedge(Y\wedge Z).$$

The maps π_1 and π_2 are continuous surjections with domain a compact space and target a Hausdorff space. Thus π_1 and π_2 are quotient maps. Since the fibres of π_1 and π_2 agree, the claim follows. $\square$

Rk 12: The map a is a homeomorphism much more generally, eg. if two out of X, Y, Z are compact Hausdorff or if X and Z are locally compact Hausdorff and Y is arbitrary. It is not a homeomorphism for arbitrary spaces, however.

Rk 13: The smash product is not the categorical product in the category of pointed spaces and pointed maps. (The categorical product is $(X,x_0)\times(Y,y_0)=(X\times Y,(x_0,y_0))$.)

Rk 14: From Exc 5, Lemma 11 and Eg. 10.(ii) and (iv) we obtain the following description of iterated suspension (at least when X is compact Hausdorff):

$$\Sigma^n X \approx X \wedge S^n \quad \text{for all } n \geq 0.$$

Exactness

We would like to show that if X is a printed compact Hausdorff space and $A \subset X$ is a closed subspace containing the basepoint, then the inclusion and quotient maps

$$A \xrightarrow{i} X \xrightarrow{q} X/A$$

induce an exact sequence

$$\tilde{K}_{\mathbb{F}}(X/A) \xrightarrow{q^*} \tilde{K}_{\mathbb{F}}(X) \xrightarrow{i^*} \tilde{K}_{\mathbb{F}}(A).$$

This will follow from the following lemma:

Lemma 15: Suppose X is a compact Hausdorff space and $A \subset X$ is a closed subspace. Let $\xi \to X$ be a vector bundle. Then

(i) A trivialization

$$h: \xi | A \xrightarrow{\approx} A \times \mathbb{F}^n$$

defines a v.b. $\xi/h \to X/A$ with the property that $\xi \approx q^*(\xi/h)$, where $q: X \to X/A$ is the quotient map.

(ii) We have $\xi/h_0 \approx \xi/h_1$ if

(a) The trivializations
$$h_0, h_1 : \xi|A \xrightarrow{\approx} A \times \mathbb{F}^n$$
are homotopic through trivializations of $\xi|A$, or

(b) h_1 is the composite
$$\xi|A \underset{\approx}{\xrightarrow{h_0}} A \times \mathbb{F}^n \xrightarrow{1 \times C} A \times \mathbb{F}^n$$
for some $C \in GL_n(\mathbb{F})$.

<u>Pf</u>: (i): Define an equivalence relation $\sim$ on ξ by declaring that $v_1 \sim v_2$ if $v_1, v_2 \in \xi|A$ and $\pi h(v_1) = \pi h(v_2)$, where $\pi : A \times \mathbb{F}^n \to \mathbb{F}^n$ is the projection. We let $\xi/h = \xi/\sim$. Then the projection $\xi \to X$ induces a map $\xi/h \to X/A$ making the square

$$\begin{array}{ccc} \xi & \xrightarrow{\tilde{q}} & \xi/h \\ \downarrow & & \downarrow \\ X & \longrightarrow & X/A \end{array}$$

($q, \tilde{q}$ the quotient maps) commutative. The fibres of $\xi/h \to X/A$ are vector spaces with $\tilde{q}$ a linear isomorphism on fibres. To prove (i), it remains to show that $\xi/h \to X/A$ is locally trivial. Away from the point $A/A \in X/A$, local triviality follows from that of $\xi | X \setminus A$. To see that $\xi/h \to X/A$ trivializes on a neighbourhood of $A/A \in X/A$, it suffices to extend the trivialisation

$$h : \xi | A \xrightarrow{\approx} A \times \mathbb{F}^n$$

to a trivialization

$$\tilde{h} : \xi | U \xrightarrow{\approx} U \times \mathbb{F}^n$$

for some neighbourhood U of A. Then $\tilde{h}$ induces a trivialisation

$$\xi/h \,|\, U/A \xrightarrow{\approx} U/A \times \mathbb{F}^n.$$

The existence of $\tilde{h}$ follows by Problem 3.5, or by the following argument: the trivialization h determines linearly independent sections

$$s_1, \ldots, s_n : A \longrightarrow \xi$$

by the formula $s_i(x) = h^{-1}(a, e_i)$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{F}^n$. Given an open $V \subset X$ s.t. $\xi | \bar{V}$ is trivial, we can use the Tietze extension theorem to extend each $s_i | A \cap \bar{V}$ to a section $\bar{V} \to \xi$. Let V_j, $j \in J$, be such open sets covering A, and for each $j \in J$ let $s_{ij} : \bar{V}_j \to \xi$ be an extension of s_i. Choose a p.o.u. $\{\varphi, \varphi_j\}_{j \in J}$ subordinate to $\{X \setminus A, V_j\}_{j \in J}$.

Then

$$\sigma_i = \sum_{j \in J} \varphi_j \, s_{ij}$$

is an extension of s_i to a section $X \to \xi$. Notice that if $\sigma_1(x), \ldots, \sigma_n(x) \in \xi_x$ are linearly independent for some $x = x_0$, they are so for all x in a neighbourhood of x_0. Thus the set

$$V = \{ x \in X \mid \sigma_1(x), \ldots, \sigma_n(x) \text{ are linearly independent} \}$$

is an open neighbourhood of A. Now the inverse of the isomorphism

$$V \times \mathbb{F}^n \xrightarrow{\approx} \xi | V$$
$$(x, (a_1, \ldots, a_n)) \longmapsto \sum_i a_i \sigma_i(x)$$

gives an extension of h.

(ii): Part (b) is clear from the construction: we have $\xi/h_0 = \xi/h_1$. In part (a), a homotopy from h_0 to h_1 through trivializations defines a trivialization

$$H : (\xi \times I) | A \times I \longrightarrow (A \times I) \times \mathbb{F}^n$$

of the v.b. $\xi \times I \longrightarrow X \times I$ agreeing with h_i over $X \times \{i\}$, $i = 0, 1$. Let

$$\pi : X/A \times I \longrightarrow X \times I / A \times I$$

be the map $\pi([x], t) = [x, t]$. Then the restriction of $\pi^*(\xi \times I / H)$ to $X/A \times \{i\}$ is the v.b. ξ/h_i, $i = 0, 1$. The claim follows. $\square$