Exactness (cont.)

Last time we proved

Lemma 1: Sp. X is a compact Hausdorff space and $A \subset X$ is a closed subspace. Let $\xi \to X$ be a vector bundle. Then

(i) A trivialization

$$h : \xi|A \xrightarrow{\approx} A \times \mathbb{F}^n$$

defines a vector bundle $\xi/h \to X/A$ with the property that $\xi \approx q^*(\xi/h)$, where $q : X \to X/A$ is the quotient map

(ii) $\xi/h_0 \approx \xi/h_1$ if

(a) $h_0, h_1 : \xi|A \xrightarrow{\approx} A \times \mathbb{F}^n$ are homotopic through trivializations of $\xi|A$, or

(b) h_1 is the composite

$$\xi|A \xrightarrow{h_0} A \times \mathbb{F}^n \xrightarrow{1 \times C} A \times \mathbb{F}^n$$

for some $C \in GL_n \mathbb{F}$. $\square$

Using lemma 1, we can prove the desired exactness result.

Prop 2: Sp. X is a pointed compact Hausdorff space and $A \subset X$ is a closed subspace containing the basepoint. Then the inclusion and quotient maps

$$A \xrightarrow{i} X \xrightarrow{q} X/A$$

induce an exact sequence

$$\tilde{K}_{\mathbb{F}}(X/A) \xrightarrow{q^*} \tilde{K}_{\mathbb{F}}(X) \xrightarrow{i^*} \tilde{K}_{\mathbb{F}}(A).$$

Pf: $\operatorname{Im} q^* \subset \operatorname{Ker} i^*$: The map qi is constant, so it factors through the one-point space. Since $\tilde{K}_{\mathbb{F}}(pt)=0$, it follows that $i^* q^* = 0$.

$\operatorname{Ker} i^* \subset \operatorname{Im} q^*$: Sp. $x \in \tilde{K}_{\mathbb{F}}(X)$ is s.t. $i^*(x)=0$. By Rk XII.1, we have $x = [\xi] - [\varepsilon^n]$ for some v.b. $\xi \to X$ and $n \geq 0$. From the assumption $i^*(x)=0$, it follows that $[\xi|A] = [\varepsilon^n] \in K_{\mathbb{F}}(A)$, so by Rk XII.3, we have $(\xi|A) \oplus \varepsilon^m \approx \varepsilon^{n+m}$ for some $m \geq 0$. Pick an isomorphism

$$h : (\xi \oplus \varepsilon^m)|A = (\xi|A) \oplus \varepsilon^m \xrightarrow{\approx} \varepsilon^{n+m}.$$

Now $[\xi \oplus \varepsilon^m/h] - [\varepsilon^{n+m}] \in \tilde{K}_{\mathbb{F}}(X/A)$ is a class with

$$\begin{aligned} & q^*\big([\xi\oplus\varepsilon^m/h] - [\varepsilon^{n+m}]\big) \\ &= [\xi \oplus \varepsilon^m] - [\varepsilon^{n+m}] \\ &= [\xi] - [\varepsilon^n] \\ &= x \in \tilde{K}_{\mathbb{F}}(X). \qquad \square \end{aligned}$$

Cor 3: If X and Y are pointed compact Hausdorff spaces, the inclusions $i_X : X \hookrightarrow X \vee Y$ and $i_Y : Y \hookrightarrow X \vee Y$ induce an isomorphism

$$\tilde{K}_{\mathbb{F}}(X \vee Y) \xrightarrow[\approx]{(i_X^*, i_Y^*)} \tilde{K}_{\mathbb{F}}(X) \times \tilde{K}_{\mathbb{F}}(Y).$$

Pf: Exercise. □

Our next goal is to use Prop 2 to construct a long exact sequence. We will need the following result.

Prop 4: Let X be a compact Hausdorff space and $A \subset X$ a contractible closed subspace. Then the map

$$q^*: \mathrm{Vect}_{\mathbb{F}}(X/A) \longrightarrow \mathrm{Vect}_{\mathbb{F}}(X)$$

induced by the quotient map $q : X \to X/A$ is a bijection.

Pf: Sp. $\xi \to X$ is a v.b. Since A is contractible, there exists a trivialization

$$h : \xi | A \xrightarrow{\approx} A \times \mathbb{F}^n.$$

Thus we get a v.b. $\xi/h \to X/A$. We claim that up to isomorphism, ξ/h is independent of the choice of h. Suppose h_0, h_1 are two trivializations of $\xi | A$. The difference between h_0 and h_1 is described by the composite

$$A \times \mathbb{F}^n \xrightarrow[\approx]{h_0^{-1}} \xi | A \xrightarrow[\approx]{h_1} A \times \mathbb{F}^n$$

which amounts to a map

$$g : A \longrightarrow GL_n \mathbb{F} .$$

Since A is contractible and hence path connected, g takes values in a single component of $GL_n \mathbb{F}$. By replacing h_1 by the composite

$$\xi | A \xrightarrow[\approx]{h_1} A \times \mathbb{F}^n \xrightarrow[\approx]{1 \times C} A \times \mathbb{F}^n$$

for a suitably chosen $C \in GL_n \mathbb{F}$ if necessary, we may assume that this component is the component $(GL_n \mathbb{F})_0$ of the identity matrix $I_n \in GL_n \mathbb{F}$. By Lemma 1.(ii).(b), this maneuver does not change the isomorphism type of ξ / h_1. Since A is contractible, the map $g : A \to (GL_n \mathbb{F})_0$ is homotopic to a constant map onto a matrix $B \in (GL_n \mathbb{F})_0$, and by continuing this homotopy by a homotopy given by a path connecting B to I_n, we obtain a homotopy from g to the constant map onto $I_n \in GL_n \mathbb{F}$. This homotopy determines a homotopy from $h_1 = (h_1 h_0^{-1}) h_0$ to $h_0 = id \circ h_0$ through trivializations of $\xi | A$, so $\xi / h_0 \approx \xi / h_1$ by Lemma 1. (ii). (a).

We have constructed a well-defined map

$$(*) \qquad \begin{aligned} Vect_{\mathbb{F}}(X) &\longrightarrow Vect_{\mathbb{F}}(X/A) \\ [\xi] &\longmapsto [\xi / h] \end{aligned}$$

For any v.b. $\xi \to X$, we have $\xi \approx q^*(\xi/h)$ by Lemma 1; and for any v.b. $\zeta \to X/A$, we have $(q^*\zeta)/h \approx \zeta$ for the evident trivialization h of $q^*\zeta|A$. Thus $(*)$ is an inverse for

$$q^* : \mathrm{Vect}_{\mathbb{F}}(X/A) \longrightarrow \mathrm{Vect}_{\mathbb{F}}(X). \quad \square$$

<u>Rk 5</u>: In favourable situations, the quotient map $q : X \to X/A$ for $A \subset X$ closed and contractible is in fact a homotopy equivalence. This happens eg. when X is a CW complex and $A \subset X$ is a contractible subcomplex.

<u>Cor 6</u>: Sp. X is a compact Hausdorff space and $A \subset X$ is a contractible closed subspace. Then the quotient map $q : X \to X/A$ induces an isomorphism

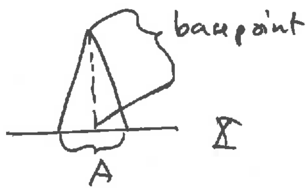
$$q^* : K_{\mathbb{F}}(X/A) \xrightarrow{\approx} K_{\mathbb{F}}(X).$$

If X has a basepoint $x_0 \in A$, then q induces an isomorphism

$$q^* : \tilde{K}_{\mathbb{F}}(X/A) \xrightarrow{\approx} \tilde{K}_{\mathbb{F}}(X). \quad \square$$

Let $i : A \to X$ be a basepoint-preserving embedding between pointed compact Hausdorff spaces, and let us use i to identify A with a subspace of X. (Notice that this subspace is closed, since A is compact and X is Hausdorff.) We can then form the union

$$X \cup CA .$$



Write

$$j(i) : X \hookrightarrow X \cup CA$$

for the inclusion;

$$\pi(i) : X \cup CA \longrightarrow X/A$$

for the map collapsing CA, and

$$q = q(i) : X \longrightarrow X/A$$

for the map collapsing A. Then

$$q(i) = \pi(i) \circ j(i) .$$

Notice that $\pi(i)^* : \tilde{K}_{\mathbb{F}}(X/A) \xrightarrow{\approx} \tilde{K}_{\mathbb{F}}(X \cup CA)$ is an iso by Cor. 6. Our goal is to show

<u>Thm 7</u>: The following sequence is exact:

$$\tilde{K}_{\mathbb{F}}(A) \xleftarrow{i^*} \tilde{K}_{\mathbb{F}}(X) \xleftarrow{q^*} \tilde{K}_{\mathbb{F}}(X/A) \xleftarrow{\delta} \tilde{K}_{\mathbb{F}}(\Sigma A) \xleftarrow{(\Sigma i)^*} \tilde{K}_{\mathbb{F}}(\Sigma X) \xleftarrow{(\Sigma q)^*} \tilde{K}_{\mathbb{F}}(\Sigma X/A) \xleftarrow{\delta} \tilde{K}_{\mathbb{F}}(\Sigma^2 A) \xleftarrow{(\Sigma^2 i)^*} \cdots$$

Here $\delta : \tilde{K}_{\mathbb{F}}(\Sigma^{n+1} A) \longrightarrow \tilde{K}_{\mathbb{F}}(\Sigma^n X/A)$ is the composite

$$\tilde{K}_{\mathbb{F}}(\Sigma^{n+1} A) \xrightarrow{(\Sigma^n q j(i))^*} \tilde{K}_{\mathbb{F}}(\Sigma^n (X \cup CA)) \xrightarrow{((\Sigma^n \pi(i))^*)^{-1}} \tilde{K}_{\mathbb{F}}(\Sigma^n X/A).$$

We start with the following observation.

<u>Lemma 8</u>: The sequence

$$A \xrightarrow{i} X \xrightarrow{j(i)} X \cup CA$$

becomes exact upon applying $\tilde{K}_{\mathbb{F}}$.

<u>Pf</u>: We have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] & X \arrow[r, "g(i)"] \arrow[dr, "j(i)"'] & X/A \\ & & X \cup CA \arrow[u, "\pi(i)"'] \end{tikzcd}$$

The top row is exact on $\tilde{K}_{\mathbb{F}}$ by Prop 2. Since $\pi(i)$ induces an isomorphism on $\tilde{K}_{\mathbb{F}}$, the claim follows. $\square$

The basic idea of the proof of Thm 7 is to apply Lemma 8 to the iterated sequence

$$A \xrightarrow{i} X \xrightarrow{j(i)} X \cup CA \xrightarrow{j^2(i)} (X \cup CA) \cup CX \longrightarrow \cdots$$

and identify the terms and maps appearing in the resulting sequence of $\tilde{K}_{\mathbb{R}}$-groups.

<u>Lemma 9</u>: The sequence

$$A \xrightarrow{i} X \xrightarrow{j(i)} X \cup CA \xrightarrow{qj(i)} \Sigma A \xrightarrow{\Sigma i} \Sigma X$$

becomes exact upon applying $\tilde{K}_{\mathbb{F}}$.

Pf: Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "i"] & X \arrow[r, "j(i)"] & X \cup CA \arrow[r, "j^2(i)"] \arrow[dr, "gj(i)"'] & (X \cup CA) \cup CX \arrow[r, "gj^2(i)"] \arrow[d, "\pi j(i)"] & \Sigma X \arrow[d, "\approx"', "-1"] \\
 & & & \Sigma A \arrow[r, "\Sigma i"] & \Sigma X
\end{tikzcd}$$

where $(-1) : \Sigma X \longrightarrow \Sigma X$ is the homeomorphism $[x,t] \longmapsto [x, 1-t]$. The triangle commutes, and the square commutes up to pointed homotopy: the map

$$h : \big((X \cup CA) \cup CX\big) \times I \longrightarrow \Sigma X$$

given by

$$h([a,s],t) = [\, i(a), 1-(1-s)t \,] \quad \text{on } CA$$
$$h([x,s],t) = [\, x, (1-s)(1-t) \,] \quad \text{on } CX$$

is a pointed homotopy from $(-1) \circ gj^2(i)$ to $\Sigma i \circ \pi j(i)$. The top row is exact on $\widetilde{K}_{\mathbb{F}}$ by Lemma 8 and Prop 2. Since $\pi j(i)$ and (-1) induce isomorphisms on $\widetilde{K}_{\mathbb{F}}$, the claim follows. □