Exactness (cont.)

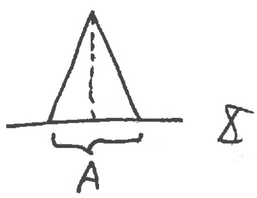
Recall from last time:

Given a basepoint-preserving embedding

$$i : A \hookrightarrow X$$

between compact Hausdorff spaces, we can form the space

$$X \cup CA$$



and consider the maps

$$j(i) : X \hookrightarrow X \cup CA \qquad \text{(inclusion)}$$

$$\pi(i) : X \cup CA \longrightarrow X/A \qquad \text{(collapse } CA \text{ — iso on } \tilde{K}_{\mathbb{F}}\text{)}$$

$$q = q(i) = \pi(i) \circ j(i) : X \longrightarrow X/A \qquad \text{(collapse } A\text{)}.$$

Our goal is to show

<u>Thm 1</u>: The following sequence is exact:

$$\begin{tikzcd} \tilde{K}_{\mathbb{F}}(A) & \tilde{K}_{\mathbb{F}}(X) \arrow[l, "i^*"'] & \tilde{K}_{\mathbb{F}}(X/A) \arrow[l, "q^*"'] \\ \tilde{K}_{\mathbb{F}}(\Sigma A) \arrow[urr, "\delta"] & \tilde{K}_{\mathbb{F}}(\Sigma X) \arrow[l, "(\Sigma i)^*"'] & \tilde{K}_{\mathbb{F}}(\Sigma X/A) \arrow[l, "(\Sigma q)^*"'] \\ \tilde{K}_{\mathbb{F}}(\Sigma^2 A) \arrow[urr, "\delta"] & \cdots \arrow[l, "(\Sigma^2 i)^*"'] & \end{tikzcd}$$

Here the maps δ are composites $\tilde{K}_{\mathbb{F}}(\Sigma^{n+1}A) \xrightarrow{(\Sigma^n \pi(i))^*} \tilde{K}_{\mathbb{F}}(\Sigma^n(X \cup CA))$
$\xrightarrow[\approx]{((\Sigma^n j(i))^*)^{-1}} \tilde{K}_{\mathbb{F}}(\Sigma^n X/A)$.

We have seen so far:

<u>Lemma 2</u>: The following sequences become exact upon applying $\tilde{K}_{\mathbb{F}}$:

(i) $A \xrightarrow{i} X \xrightarrow{q(i)} X/A$

(ii) $A \xrightarrow{i} X \xrightarrow{j(i)} X \cup CA$

(iii) $A \xrightarrow{i} X \xrightarrow{j(i)} X \cup CA \xrightarrow{gj(i)} \Sigma A \xrightarrow{\Sigma i} \Sigma X$. $\square$

Lemma 2. (iii) implies

<u>Lemma 3</u>: The sequence

$$\Sigma A \xrightarrow{\Sigma i} \Sigma X \xrightarrow{\Sigma j(i)} \Sigma(X \cup CA) \xrightarrow{\Sigma gj(i)} \Sigma^2 A \xrightarrow{\Sigma^2 i} \Sigma^2 X$$

becomes exact upon applying $\tilde{K}_{\mathbb{F}}$.

<u>Pf</u>: We have a commutative diagram

$$\begin{tikzcd}
\Sigma A \arrow[r, "\Sigma i"] & \Sigma X \arrow[r, "j(\Sigma i)"] \arrow[dr, "\Sigma j(i)"'] & \Sigma X \cup C\Sigma A \arrow[r, "gj(\Sigma i)"] \arrow[d, "\approx"', "\alpha"] & \Sigma^2 A \arrow[r, "\Sigma^2 i"] \arrow[d, "\approx"', "\beta"] & \Sigma^2 X \arrow[d, "\approx"', "\gamma"] \\
 & & \Sigma(X \cup CA) \arrow[r, "\Sigma gj(i)"] & \Sigma^2 A \arrow[r, "\Sigma^2 i"] & \Sigma^2 X
\end{tikzcd}$$

where α, β, γ are the homeomorphisms defined by

$\alpha([x,t]) = [x,t]$ on ΣX

$\alpha([a,t,s]) = [a,s,t]$ on $C\Sigma A$ (s = cone coordinate, t = suspension coordinate)

$\beta([a,t,s]) = [a,s,t]$

$\gamma([x,t,s]) = [x,s,t]$.

The top row is exact on $\tilde{K}_{\mathbb{F}}$ by Lemma 2.(iii). The claim follows. $\square$

Pf of Thm 1: We have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] & X \arrow[r, "j(i)"] \arrow[dr, "q"'] & X \cup CA \arrow[r, "qj(i)"] \arrow[d, "\pi(i)"] & \Sigma A \arrow[r, "\Sigma i"] & \Sigma X \arrow[r, "\Sigma j(i)"] \arrow[dr, "\Sigma q"'] & \Sigma(X \cup CA) \arrow[r, "\Sigma qj(i)"] \arrow[d, "\Sigma\pi(i)"] & \Sigma^2 A \arrow[r, "\Sigma^2 i"] & \cdots \\ & & X/A & & & \Sigma X/A & & \end{tikzcd}$$

where the top row is exact on $\tilde{K}_{\mathbb{F}}$ by Lemmas 2.(iii) and 3. Now notice that for each n, the map $\Sigma^n \pi(i)$ fits into a commutative square

$$\begin{tikzcd} \Sigma^n(X \cup CA) \arrow[d, "\Sigma^n \pi(i)"'] & \Sigma^n X \cup C\Sigma^n A \arrow[l, "\alpha"', "\approx"] \arrow[d] \\ \Sigma^n X/A & \Sigma^n X/\Sigma^n A \arrow[l, "\beta"', "\approx"] \end{tikzcd}$$

where α and β are the homeomorphisms given by

$$\alpha([x, t_1, \ldots, t_n]) = [x, t_1, \ldots, t_n] \quad \text{on } \Sigma^n X$$

$$\alpha([x, t_1, \ldots, t_n, s]) = [x, s, t_1, \ldots, t_n] \quad \text{on } C\Sigma^n A$$

$$\beta([x, t_1, \ldots, t_n]) = [[x], t_1, \ldots, t_n].$$

Since $\pi(\Sigma^n i)$ is an isomorphism on $\tilde{K}_{\mathbb{F}}$, it follows that $\Sigma^n \pi(i)$ also is. The claim follows. □

Negative K-groups

Thm 1 motivates the following definition.

Def 4: Let $n \geq 0$. We set

$$\tilde{K}^{-n}_{\mathbb{F}}(X) := \tilde{K}_{\mathbb{F}}(\Sigma^n X) \qquad (X \text{ pointed compact Hausdorff})$$
$$K^{-n}_{\mathbb{F}}(X) := \tilde{K}^{-n}_{\mathbb{F}}(X_+) \qquad (X \text{ compact Hausdorff})$$
$$K^{-n}_{\mathbb{F}}(X,A) := \tilde{K}^{-n}_{\mathbb{F}}(X/A) \qquad ((X,A) \text{ compact pair})$$

If $f : X \to Y$ is a pointed map, we define

$$f^*: \tilde{K}^{-n}_{\mathbb{F}}(Y) \longrightarrow \tilde{K}^{-n}_{\mathbb{F}}(X)$$

to be the map

$$(\Sigma^n f)^* : \tilde{K}_{\mathbb{F}}(\Sigma^n Y) \longrightarrow \tilde{K}_{\mathbb{F}}(\Sigma^n X).$$

For any ~~...~~ continuous map $f : X \to Y$, we now define

$$f^* : K^{-n}_{\mathbb{F}}(Y) \longrightarrow K^{-n}_{\mathbb{F}}(X)$$

to be the map induced by the evident pointed map

$$f_+ : X_+ \longrightarrow Y_+ .$$

For a map of pairs $f : (X,A) \to (Y,B)$, we define

$$f^* : K^{-n}_{\mathbb{F}}(Y,B) \longrightarrow K^{-n}_{\mathbb{F}}(X,A)$$

to be the map induced by the map

$$X/A \longrightarrow Y/B$$

defined by f.

Rk 5: Observe that for $n=0$,

$$\tilde{K}^0_{\mathbb{F}}(X) = \tilde{K}_{\mathbb{F}}(X)$$
$$K^0_{\mathbb{F}}(X) = K_{\mathbb{F}}(X)$$
$$K^0_{\mathbb{F}}(X,A) = K_{\mathbb{F}}(X,A)$$

and the induced maps on $\tilde{K}^0_{\mathbb{F}}$, $K^0_{\mathbb{F}}$ and relative $K^0_{\mathbb{F}}$ agree with those for $\tilde{K}_{\mathbb{F}}$, $K_{\mathbb{F}}$ and relative $K_{\mathbb{F}}$, respectively.

From Thm 1, we get long exact sequences

$$\tilde{K}^0_{\mathbb{F}}(A) \xleftarrow{i^*} \tilde{K}^0_{\mathbb{F}}(X) \xleftarrow{q^*} \tilde{K}^0_{\mathbb{F}}(X/A) \xleftarrow{\delta} \tilde{K}^{-1}_{\mathbb{F}}(A) \xleftarrow{i^*} \tilde{K}^{-1}_{\mathbb{F}}(X) \xleftarrow{q^*} \cdots$$

(X pointed compact Hausdorff, $A \subset X$ closed subspace containing the basepoint, $i: A \hookrightarrow X$ inclusion, $q: X \to X/A$ the quotient map) and

$$K^0_{\mathbb{F}}(A) \xleftarrow{i^*} K^0_{\mathbb{F}}(X) \xleftarrow{j^*} K^0_{\mathbb{F}}(X,A) \xleftarrow{\delta} K^{-1}_{\mathbb{F}}(A) \xleftarrow{i^*} K^{-1}_{\mathbb{F}}(X) \xleftarrow{j^*} \cdots$$

((X,A) compact pair, $i: A \hookrightarrow X$ and $j: X = (X,\emptyset) \hookrightarrow (X,A)$ the inclusions).

Write
$$\tilde{K}^{\leq 0}_{\mathbb{F}}(X) := \bigoplus_{n \geq 0} \tilde{K}^{-n}_{\mathbb{F}}(X)$$
(and similarly for unreduced and relative K-groups).
We regard $\tilde{K}^{\leq 0}_{\mathbb{F}}(X)$ as a $\mathbb{Z}_{\leq 0}$-graded abelian group with degree $(-n)$-part given by $\tilde{K}^{-n}_{\mathbb{F}}(X)$.

<u>Recall / Def $\overset{6}{7}$</u>: Sp. $(A,+)$ is a commutative monoid; relevant examples include $(\mathbb{Z},+)$, $(\mathbb{Z}_{\geq 0},+)$, $(\mathbb{Z}_{\leq 0},+)$ and $(\mathbb{Z}/2,+)$. An <u>A-graded abelian group</u> is an abelian group M^* together with a direct sum decomposition

$$M^* = \bigoplus_{a\in A} M^a.$$

The elements of $M^a \subset M^*$ are the <u>homogeneous elements of degree</u> a in M^*. A <u>homomorphism</u>

$$f : M^* \longrightarrow N^*$$

of A-graded abelian groups is a family of morphisms

$$f = f^a : M^a \longrightarrow N^a,$$

one for each $a \in A$; equivalently, f is a homomorphism between the underlying abelian groups of M^* and N^* satisfying $f(M^a) \subset N^a$ for all $a \in A$. The <u>direct sum</u> of graded abelian groups M^* and N^* is the graded ab. group ~~[illegible]~~ $M^* \oplus N^*$ with degree a part $M^a \oplus N^a$. The <u>tensor product</u> of M^* and N^* is the graded ab. group $M^* \otimes N^*$ with degree a part

$$\bigoplus_{\substack{b,c\in A \\ b+c=a}} M^b \otimes N^c.$$

If M is an abelian group, we can regard it as the graded abelian group M^* with $M^0 = M$ and $M^a = 0$ for $a \in A\setminus\{0\}$.

A <u>non-unital graded ring</u> is a graded abelian group R^* together with an associative product

$$\mu : R^* \otimes R^* \longrightarrow R^*;$$

equivalently, it is an ordinary non-unital ring equipped with the structure of a graded abelian group s.t. $R^a \cdot R^b \subset R^{a+b}$ for all $a, b \in A$. R^* is <u>unital</u> if there exists $1 \in R^0$ s.t. $1 \cdot x = x \cdot 1 = x$ for all $x \in R^*$. Suppose A is equipped with the notion of parity, i.e. a homomorphism $(A, +) \longrightarrow (\mathbb{Z}/2, +)$; we say that $a \in A$ is even if $\varepsilon(a) = 0$ and odd if $\varepsilon(a) = 1$. Then R^* is <u>graded commutative</u> (or just <u>commutative</u>) if for all homogeneous elements $x, y \in R^*$ we have

$$x \cdot y = (-1)^{\varepsilon(|x|) \cdot \varepsilon(|y|)} y \cdot x,$$

where we have written $|\cdot|$ for the degree of a homogeneous element. Notice that all of the aforementioned examples of A come equipped with an ε, with the usual notion of parity.

<u>Eg</u>: For any $d \in \mathbb{Z}$, the polynomial ring $\mathbb{Z}[x]$ becomes a $\mathbb{Z}$-graded ring (with unit) if we assign x the degree d. It is graded commutative if d is even but not if d is odd.

Products

Def 7: For X, Y compact Hausdorff, define the *external product*

$$K_{\mathbb{F}}(X) \otimes K_{\mathbb{F}}(Y) \xrightarrow{\;*\;} K_{\mathbb{F}}(X \times Y)$$

by $a * b = \pi_X^*(a)\,\pi_Y^*(b)$ where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are the projections.

Notice that the external product is natural in the sense that if $f : X \to X'$ and $g : Y \to Y'$, then $f^*(a) * g^*(b) = (f \times g)^*(a \times b)$.

We would like to define a reduced version of the external product in the case where X and Y are pointed.

From the commutativity of the square

$$\begin{array}{ccc}
K_{\mathbb{F}}(X) \otimes K_{\mathbb{F}}(Y) & \xrightarrow{\;*\;} & K_{\mathbb{F}}(X \times Y) \\
\Big\downarrow {\scriptstyle i_{x_0}^* \otimes i_{y_0}^*} & & \Big\downarrow {\scriptstyle i_{(x_0,y_0)}^*} \\
K_{\mathbb{F}}(x_0) \otimes K_{\mathbb{F}}(y_0) & \xrightarrow{\;*\;} & K_{\mathbb{F}}((x_0,y_0))
\end{array}$$

(where the vertical maps are induced by inclusions of basepoints) we deduce that if

$$a \in \tilde{K}_{\mathbb{F}}(X) = \operatorname{Ker}(i_{x_0}^*) \quad \text{and}$$
$$b \in \tilde{K}_{\mathbb{F}}(Y) = \operatorname{Ker}(i_{y_0}^*)$$

then $a * b \in \tilde{K}_{\mathbb{F}}(X \times Y) = \operatorname{Ker}(i_{(x_0,y_0)}^*)$.

The maps

$$X \vee Y \hookrightarrow X \times Y \xrightarrow{q} X \wedge Y$$

induce an exact sequence

$$\begin{array}{ccccccccc}
\tilde{K}_{\mathbb{F}}(\Sigma(X\times Y)) & \longrightarrow & \tilde{K}_{\mathbb{F}}(\Sigma(X\vee Y)) & \longrightarrow & \tilde{K}_{\mathbb{F}}(X\wedge Y) & \xrightarrow{q^*} & \tilde{K}_{\mathbb{F}}(X\times Y) & \longrightarrow & \tilde{K}_{\mathbb{F}}(X\vee Y) \\
 & & \wr\wr & & & & & & \wr\wr \\
 & & \tilde{K}_{\mathbb{F}}(\Sigma X \vee \Sigma Y) & & & & & & \tilde{K}_{\mathbb{F}}(X)\oplus\tilde{K}_{\mathbb{F}}(Y) \\
 & & \wr\wr & & & & & & \\
 & & \tilde{K}_{\mathbb{F}}(\Sigma X)\oplus\tilde{K}_{\mathbb{F}}(\Sigma Y) & & & & & &
\end{array}$$

The first map in the sequence has a section given by

$$(\Sigma\pi_X)^* + (\Sigma\pi_Y)^* : \tilde{K}_{\mathbb{F}}(\Sigma X)\oplus\tilde{K}_{\mathbb{F}}(\Sigma Y) \longrightarrow \tilde{K}_{\mathbb{F}}(\Sigma(X\times Y))$$

and the last map has a section given by

$$\pi_X^* + \pi_Y^* : \tilde{K}_{\mathbb{F}}(X)\oplus\tilde{K}_{\mathbb{F}}(Y) \longrightarrow \tilde{K}_{\mathbb{F}}(X\vee Y).$$

Thus we get a split short exact sequence

$$0 \to \tilde{K}_{\mathbb{F}}(X\wedge Y) \xrightarrow{q^*} \tilde{K}_{\mathbb{F}}(X\times Y) \xrightarrow{(i_X^*,\, i_Y^*)} \tilde{K}_{\mathbb{F}}(X)\oplus\tilde{K}_{\mathbb{F}}(Y) \to 0$$

where i_X and i_Y are the inclusions

$$i_X : X \to X\times Y,\quad x \mapsto (x, y_0) \qquad\qquad i_Y : Y \to X\times Y,\quad y \mapsto (x_0, y).$$

If $a\in\tilde{K}_{\mathbb{F}}(X)$ and $b\in\tilde{K}_{\mathbb{F}}(Y)$, then

$$a*b = \pi_X^*(a)\,\pi_Y^*(b),$$

and $i_Y^*\pi_X^*(a) = 0$ and $i_X^*\pi_Y^*(b)=0$, so $(i_X^*, i_Y^*)(a*b) = 0$ and $a*b$ pulls back along q^* to a unique

Recall that $K_{\mathbb{F}}(X) \approx \tilde{K}_{\mathbb{F}}(X) \oplus \mathbb{Z}$; more precisely, the map

$$\begin{aligned} \tilde{K}_{\mathbb{F}}(X) \oplus \mathbb{Z} &\longrightarrow K_{\mathbb{F}}(X) \\ (a,n) &\longmapsto a+n \end{aligned}$$

is an isomorphism. (Here $n \in K(X)$ means the image of $n \in \mathbb{Z}$ under the unique ring homomorphism $\mathbb{Z} \to K(X)$; concretely, $n = [\varepsilon^n]$ if $n \geq 0$ and $n = -[\varepsilon^{-n}]$ if $n < 0$.)

<u>Lemma 8</u>: The reduced and unreduced external products fit into a commutative square

$$\begin{array}{ccc} K_{\mathbb{F}}(X) \otimes K_{\mathbb{F}}(Y) & \xleftarrow{\approx} & (\tilde{K}_{\mathbb{F}}(X) \otimes \tilde{K}_{\mathbb{F}}(Y)) \oplus \tilde{K}_{\mathbb{F}}(X) \oplus \tilde{K}_{\mathbb{F}}(Y) \oplus \mathbb{Z} \\ \Big\downarrow * & & \Big\downarrow * \oplus \mathrm{id} \oplus \mathrm{id} \oplus \mathrm{id} \\ K_{\mathbb{F}}(X \times Y) & \xleftarrow{\approx} & \tilde{K}_{\mathbb{F}}(X \wedge Y) \oplus \tilde{K}_{\mathbb{F}}(X) \oplus \tilde{K}_{\mathbb{F}}(Y) \oplus \mathbb{Z} \end{array}$$

Here the top isomorphism is induced by the isomorphisms

$$\tilde{K}_{\mathbb{F}}(X) \oplus \mathbb{Z} \xrightarrow{\approx} K_{\mathbb{F}}(X) \quad \text{and} \quad \tilde{K}_{\mathbb{F}}(Y) \oplus \mathbb{Z} \xrightarrow{\approx} K_{\mathbb{F}}(Y)$$

while the bottom isomorphism is induced by the isomorphisms

$$\tilde{K}_{\mathbb{F}}(X \wedge Y) \oplus \tilde{K}_{\mathbb{F}}(X) \oplus \tilde{K}_{\mathbb{F}}(Y) \xrightarrow[\approx]{q^* + \pi_X^* + \pi_Y^*} \tilde{K}_{\mathbb{F}}(X \times Y)$$

(why is this an isomorphism?) and

$$\tilde{K}_{\mathbb{F}}(X \times Y) \oplus \mathbb{Z} \xrightarrow{\approx} K_{\mathbb{F}}(X \times Y).$$

<u>Pf</u>: Exercise. $\square$

We now generalize the external products to negative K-groups.

<u>Def 9</u>: Sp. X, Y are pointed compact Hausdorff spaces. For all $n, m \geq 0$, we define the <u>reduced</u> <u>external</u> <u>product</u>

$$\tilde{K}^{-n}_{\mathbb{F}}(X) \otimes \tilde{K}^{-m}_{\mathbb{F}}(Y) \xrightarrow{*} \tilde{K}^{-n-m}_{\mathbb{F}}(X \wedge Y)$$

as the composite

$$\begin{aligned}\tilde{K}_{\mathbb{F}}(X \wedge S^n) \otimes \tilde{K}_{\mathbb{F}}(Y \wedge S^m) &\xrightarrow{*} \tilde{K}_{\mathbb{F}}(X \wedge S^n \wedge Y \wedge S^m) \\ &\xrightarrow[(1 \wedge T \wedge 1)^*]{\approx} \tilde{K}_{\mathbb{F}}(X \wedge Y \wedge S^n \wedge S^m) \\ &\xrightarrow{\approx} \tilde{K}_{\mathbb{F}}(X \wedge Y \wedge S^{n+m}).\end{aligned}$$

Here $T : Y \wedge S^n \to S^n \wedge Y$ is the homeomorphism $T(y \wedge t) = t \wedge y$, and we have used the homeomorphism $\Sigma^k Z \approx Z \wedge S^k$. Replacing X by X_+ and Y by Y_+, we get the <u>unreduced</u> <u>external</u> <u>product</u>

$$K^{-n}_{\mathbb{F}}(X) \otimes K^{-m}_{\mathbb{F}}(Y) \xrightarrow{*} K^{-n-m}_{\mathbb{F}}(X \times Y).$$