Bott periodicity in complex K-theory

There are several equivalent formulations of the Bott periodicity theorem, some of which do not explicitly involve K-theory. Here are some K-theoretical formulations of the complex Bott periodicity theorem:

Thm 1 (complex Bott periodicity):

Version 1: The map

$$K(X) \otimes K(S^2) \xrightarrow{*} K(X \times S^2)$$

is an isomorphism and $K(S^2)$ is the free abelian group generated by 1 and $[H]$. (Recall: H is the dual of the canonical line bundle $\gamma^1 \to \mathbb{C}P^1 = S^2$.) Here X is any compact Hausdorff space.

Version 2: The map

$$\tilde{K}(X) \otimes \tilde{K}(S^2) \xrightarrow{*} \tilde{K}(X \wedge S^2)$$

is an isomorphism and $\tilde{K}(S^2)$ is the free abelian group generated by the "Bott element" $b = [H] - 1$. Here X is any pointed compact Hausdorff space.

Version 3: The map

$$\beta : \tilde{K}(X) \longrightarrow \tilde{K}(X \wedge S^2)$$

given by multiplication with the Bott element $b \in \tilde{K}(S^2)$ is an isomorphism. Here X is any pointed compact Hausdorff space.

Version 4: The map
$$\beta : \tilde{K}(X) \longrightarrow \tilde{K}^{-2}(X)$$
given by multiplication by the Bott element $b \in \tilde{K}(S^2) = \tilde{K}^{-2}(S^0)$ is an isomorphism. Here X is any pointed compact Hausdorff space.

Version 5: For all $n \geq 0$, the map
$$\beta : \tilde{K}^{-n}(X) \longrightarrow \tilde{K}^{-n-2}(X)$$
given by multiplication by the Bott element $b \in \tilde{K}(S^2) = \tilde{K}^{-2}(S^0)$ is an isomorphism. Here X is any pointed compact Hausdorff space.

Prop 2: The aforementioned versions of the Bott periodicity theorem are equivalent.

Pf: $v1 \Leftrightarrow v2$ by Lemma XVI.8.
$v2 \Rightarrow v3$ is immediate.
The statement about $\tilde{K}(S^2)$ in $v2$ follows from $v3$ by taking $X = S^0$. Then $v3 \Rightarrow v2$ is clear.
$v3 \Leftrightarrow v4$ is immediate.
$v4$ is a special case of $v5$, and $v5$ follows from $v3$ by unrolling the definition of the reduced external product. $\square$

Rk 3: There is an analogous theorem in real K-theory, but with S^8 in place of S^2. The proof of the real version is more complicated.

There are many different proofs of the periodicity theorem from a wide variety of different perspectives. Bott's original proof [R. Bott, The stable homotopy of the classical groups. Annals of Mathematics (2) Vol. 70 (1959) pp. 313 - 337] was based on Morse theory. We will discuss a variant of the "elementary" K-theoretical proof given by Atiyah and Bott [M. Atiyah and R. Bott, On the periodicity theorem for complex vector bundles. Acta Mathematica 112 (1964) pp. 229 - 247]. In the exposition, we will draw from Husemöller's book [D. Husemöller, Fibre Bundles. Graduate Texts in Mathematics 20]. The basic idea of the proof is to carefully analyze complex vector bundles over $X \times S^2$ in terms of clutching functions.

Let us start the proof by establishing notation. Interpret

$$S^2 = \mathbb{C} \cup \{\infty\} \qquad \text{(Riemann sphere)}$$

and let

$$D_0^2 = \{ z \in S^2 \mid |z| \leq 1 \}$$

$$D_\infty^2 = \{ z \in S^2 \mid |z| \geq 1 \} \qquad (|\infty| = \infty).$$

Then $D_0^2 \cup D_\infty^2 = S^2$ and $D_0^2 \cap D_\infty^2 = S^1 \subset \mathbb{C}$.
Suppose X is a compact Hausdorff space. Then

$$X \times S^2 = X \times D_0^2 \cup X \times D_\infty^2$$

with

$$X \times D_0^2 \cap X \times D_\infty^2 = X \times S^1 .$$

Write

$$\pi_0 : X \times D^2_0 \longrightarrow X$$

$$\pi_\infty : X \times D^2_\infty \longrightarrow X$$

$$\pi : X \times S^1 \longrightarrow X$$

for the projections and

$$s : X \longrightarrow X \times S^2$$

for the embedding $s(x) = (x, 1)$.

By the following proposition, every complex v.b. $\xi \to X \times S^2$ is of the form $\pi_0^* \varsigma \cup_u \pi_\infty^* \varsigma$ for some complex v.b. $\varsigma \to X$ and clutching function $u : \pi^* \varsigma \xrightarrow{\approx} \pi^* \varsigma$.

<u>Prop 4</u>: To simplify notation, identify $X \times \{1\} \subset X \times S^2$ with X via the embedding s. Let $\xi \to X \times S^2$ be a complex v.b. and let $\varsigma = \xi | X$. Then $\exists$ isomorphism

$$u : \pi^* \varsigma \xrightarrow{\approx} \pi^* \varsigma$$

of v.b.'s over $X \times S^1$ with the following properties:

1) u is the identity map over $X \subset X \times S^1$. Notice that then the restriction $(\pi_0^* \varsigma \cup_u \pi_\infty^* \varsigma) | X$ can be canonically identified with ς.

2) $\exists$ isomorphism $\xi \xrightarrow{\approx} \pi_0^* \varsigma \cup_u \pi_\infty^* \varsigma$ which is the identity over $X \subset X \times S^2$.

Moreover, 1) and 2) determine u uniquely up to homotopy (through v.b. isomorphisms).

Pf: The maps

$$X \times D^2_0 \xrightarrow{\pi_0} X \hookrightarrow X \times D^2_0$$
$$X \times D^2_\infty \xrightarrow{\pi_\infty} X \hookrightarrow X \times D^2_\infty$$

are homotopic to the identity maps, so $\exists$ isomorphisms

$$h_0 : \xi \mid X \times D^2_0 \xrightarrow{\approx} \pi_0^* \zeta$$
$$h_\infty : \xi \mid X \times D^2_\infty \xrightarrow{\approx} \pi_\infty^* \zeta .$$

By composing h_0 with a suitable automorphism of $\pi_0^* \zeta = \zeta \times D^2_0$ of the form $\alpha \times id_{D^2_0}$, $\alpha : \zeta \xrightarrow{\approx} \zeta$, we may assume that h_0 is the identity over X, and similarly for h_∞. Now the composite

$$u : \pi^* \zeta \underset{\approx}{\xrightarrow{h_0^{-1}|}} \xi \mid X \times S^1 \underset{\approx}{\xrightarrow{h_\infty |}} \pi^* \zeta$$

is as desired. To see the uniqueness of u, sp. that $u' : \pi^* \zeta \xrightarrow{\approx} \pi^* \zeta$ is another isomorphism satisfying 1) and 2). From the isomorphism $\xi \xrightarrow{\approx} \pi_0^* \zeta \cup_{u'} \pi_\infty^* \zeta$ of part 2), we get isomorphisms

$$h_0' : \xi \mid X \times D^2_0 \xrightarrow{\approx} \pi_0^* \zeta$$
$$h_\infty' : \xi \mid X \times D^2_\infty \xrightarrow{\approx} \pi_\infty^* \zeta$$

restricting to the identity over X and s.t. u' agrees with the composite

$$\pi^* \zeta \underset{\approx}{\xrightarrow{(h_0')^{-1}|}} \xi \mid X \times S^1 \underset{\approx}{\xrightarrow{h_\infty' |}} \pi^* \zeta .$$

The difference between h_0 and h_0' is an automorphism δ of $\pi_0^* \zeta = \zeta \times D^2_0$ which is the identity over X.

Such an automorphism is of the form

$$\xi \times D^2_0 \xrightarrow{\approx} \xi \times D^2_0$$
$$(v, z) \longmapsto (g(v,z), z)$$

where for each $z \in D^2_0$, the map $g(-,z) : \xi \longrightarrow \xi$ is a v.b. isomorphism and $g(-,1) = id_\xi$. Now a deformation retraction H from D^2_0 onto $1 \in D^2_0$ gives a homotopy

$$\xi \times D^2_0 \times I \longrightarrow \xi \times D^2_0$$
$$(v, z, t) \longmapsto (g(v, H(z,t)), z)$$

from δ to $id_{\pi_0^* \xi}$ and hence a homotopy between h_0 and h_0' (through v.b. isomorphisms). Similarly, h_∞ and h_∞' are homotopic through v.b. isomorphisms. There results a homotopy between $u = (h_\infty|) \circ (h_0^{-1}|)$ and $u' = (h_\infty'|) \circ ((h_0')^{-1}|)$ through v.b. isomorphisms. □

· If $\xi \longrightarrow X$ is a v.b., an automorphism

$$u : \pi^* \xi \xrightarrow{\approx} \pi^* \xi$$

is a continuous family of automorphisms

$$u(x,z) : \xi_x \longrightarrow \xi_x$$

for $x \in X$, $z \in S^1$.

Def 5: Let $\xi \to X$ be a complex v.b. An automorphism $u : \pi^* \xi \longrightarrow \pi^* \xi$ is called a Laurent polynomial (resp. polynomial) clutching function if it is of the form

$$u(x,z) = \sum_{|k|\le n} a_k(x) z^k \qquad (\text{resp. } u(x,z) = \sum_{0\le k\le n} a_k(x) z^k)$$

for some endomorphisms $a_k : \xi \to \xi$ of ξ. It is called a <u>linear clutching function</u> if

$$u(x,z) = a(x) + b(x) z$$

for some endomorphisms $a, b : \xi \to \xi$.

Our next goal is to show that any clutching function can be approximated by a Laurent polynomial clutching function.

Suppose $\xi \to X$ is a complex vector bundle. Write $\mathrm{End}(\pi^*\xi)$ for the vector space of vector bundle endomorphisms of $\pi^*\xi \to X \times S^1$. Pick a Hermitian metric on ξ and give $\pi^*\xi$ the induced metric. Then

$$\|\alpha\| = \sup_{\substack{v\in\pi^*\xi \\ \|v\|\le 1}} \|\alpha(v)\|$$

defines a norm on $\mathrm{End}(\pi^*\xi)$. The norm satisfies the triangle inequality, so balls in $\mathrm{End}(\pi^*\xi)$ are convex. The set $\mathrm{Aut}(\pi^*\xi) \subset \mathrm{End}(\pi^*\xi)$ of automorphisms of $\pi^*\xi$ is open in $\mathrm{End}(\pi^*\xi)$, since it is the preimage of $(0,\infty)$ under the continuous map

$$\begin{aligned} \mathrm{End}(\pi^*\xi) &\longrightarrow \mathbb{R}_{\ge 0} \\ \alpha &\longmapsto \inf_{(x,z)\in X\times S^1} |\det(\alpha(x,z))| . \end{aligned}$$

It follows that if $u : \pi^*\xi \xrightarrow{\cong} \pi^*\xi$ is a v.b. automorphism, there is an $\varepsilon > 0$ s.t.

$$B_\varepsilon(u) \subset \mathrm{Aut}(\pi^* \zeta)$$

$\uparrow$ the open ε-ball around u

and that for such $\varepsilon > 0$, all $u' \in B_\varepsilon(u)$ are homotopic through v.b. isomorphisms.

Prop 6: Laurent polynomial clutching functions are dense in $\mathrm{Aut}(\pi^* \zeta)$.

Pf: Next time. $\square$

Cor 7: For any clutching function $u : \pi^*\zeta \xrightarrow{\approx} \pi^*\zeta$, there exists a Laurent polynomial clutching function $v : \pi^*\zeta \xrightarrow{\approx} \pi^*\zeta$ homotopic to it. $\square$