Bott periodicity in complex K-theory (cont.)

Recall: We are trying to prove the Bott periodicity theorem in complex K-theory, one formulation of which is

<u>Thm 1</u>: Let X be a pointed compact Hausdorff space. Then the map

$$\tilde{K}(X) \xrightarrow{\beta} \tilde{K}(X \wedge S^2)$$

given by reduced external product with the Bott class $b = [H] - 1 \in \tilde{K}(S^2)$ is an isomorphism.

<u>Recall</u>: 1) If $\xi \to X$ is a complex v.b., an automorphism $u : \pi^*\xi \xrightarrow{\cong} \pi^*\xi$ ($\pi : X \times S^1 \to X$ the projection) is called a <u>Laurent polynomial clutching function</u> if it is of the form

$$u(x,z) = \sum_{|k| \le n} a_k(x) z^k$$

for some endomorphisms $a_k : \xi \to \xi$.

2) Choosing a Hermitian metric on ξ, we obtained a norm

$$\|\alpha\| = \sup_{\substack{v \in \pi^*\xi \\ \|v\| \le 1}} \|\alpha(v)\|$$

on the vector space $\mathrm{End}(\pi^*\xi)$ of vector bundle endomorphisms of $\pi^*\xi$. The set $\mathrm{Aut}(\pi^*\xi)$ of vector bundle automorphisms of $\pi^*\xi$ is open in $\mathrm{End}(\pi^*\xi)$.

Prop 2: The Laurent polynomial clutching functions are dense in $\mathrm{Aut}(\pi^*\xi)$.

Sketch of pf: Given $u: \pi^*\xi \xrightarrow{\approx} \pi^*\xi$, define endomorphisms $a_k: \xi \to \xi$ by

$$a_k(x) := \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\varphi} u(x, e^{i\varphi})\, d\varphi$$

and define

$$s_k(x,z) := \sum_{|j| \le k} a_j(x) z^j$$

$$u_n(x,z) := \frac{1}{n+1} \sum_{0 \le k \le n} s_k(x,z)$$

(so $u_n(x,z)$ is the n-th partial Cesaro sum of a Fourier series). A generalization of Fejér's theorem implies that

$$u_n(x,z) \xrightarrow{n \to \infty} u(x,z)$$

uniformly in x and z. The claim follows.

Alternatively, see Prop. 2.4 in Hatcher's draft for a book for a detailed alternative argument. □

Cor 3: (i) Any clutching function $u: \pi^*\xi \xrightarrow{\approx} \pi^*\xi$ is homotopic through clutching functions to a Laurent polynomial clutching function.
(ii) If $l_0, l_1: \pi^*\xi \to \pi^*\xi$ are Laurent polynomial clutching functions which are homotopic through clutching functions, then l_0 and l_1 are homotopic through Laurent polynomial clutching functions.

Pf: (i): If l is a Laurent polynomial clutching function sufficiently close to u, then the linear homotopy $(1-t)u + tl$ is a homotopy from u to l through clutching functions.

(ii): A homotopy from l_0 to l_1 through clutching fn's defines an automorphism of the v.b. $\mathcal{E}'_S \times I \to \mathcal{E} \times I$. Approximating this automorphism by a Laurent polynomial clutching fn, we obtain a homotopy l'_t through Laurent polynomial clutching fn's. As long as the approximation is close enough, we can combine l'_t with the linear homotopies from l_0 to l'_0 and l'_1 to l_1 to get a homotopy from l_0 to l_1 through Laurent polynomial clutching fn's. $\square$

Reduction to finding α

Instead of simply following Husemöller's exposition of the Atiyah - Bott proof of the periodicity theorem, as I originally planned, I now plan to take a shortcut which makes the proof more efficient. The proof will be centered around the construction of the map α of the following proposition. In this section, our goal is to show how the existence of α implies the periodicity theorem.

<u>Prop 4</u>: For every compact Hausdorff space X, there exists a map

$$\alpha : K(X \times S^2) \longrightarrow K(X)$$

with the following properties:

1) α is natural in X (in the sense that for all $f : X \to Y$, the square

$$\begin{array}{ccc} K(Y \times S^2) & \xrightarrow{\alpha} & K(Y) \\ \Big\downarrow{\scriptstyle (f \times id_{S^2})^*} & & \Big\downarrow{\scriptstyle f^*} \\ K(X \times S^2) & \xrightarrow{\alpha} & K(X) \end{array}$$

commutes).

2) α is a homomorphism of $K(X)$-modules (where the $K(X)$-module structure on $K(X \times S^2)$ is given by $a \cdot x = \pi_X^*(a)\, x$ for $a \in K(X)$ and $x \in K(X \times S^2)$; here $\pi_X : X \times S^2 \to X$ is the projection).

3) For $X = pt$ and $b = [H] - 1 \in K(S^2)$ the Bott element, $\alpha(b) = 1 \in K(pt)$.

We will now show how the periodicity theorem follows from Prop. 4. Let α be as in Prop 4. For X a pointed compact Hausdorff space, we define a reduced version

$$\alpha : \tilde{K}(X \wedge S^2) \longrightarrow \tilde{K}(X)$$

of α as follows:

Let $x_0 \in X$ be the basepoint and write

$$i : \{x_0\} \hookrightarrow X \quad \text{and} \quad j : \{x_0\} \times S^2 \hookrightarrow X \times S^2$$

for the inclusions. We then have the diagram

$$\begin{tikzcd}
 & \tilde{K}(X \times S^2/\{x_0\} \times S^2) \arrow[r] \arrow[d, dashed, "\alpha"'] & K(X \times S^2) \arrow[r, "j^*"] \arrow[d, "\alpha"] & K(\{x_0\} \times S^2) \arrow[d, "\alpha"] & \\
0 \arrow[r] & \tilde{K}(X) \arrow[r] & K(X) \arrow[r, "i^*"] & K(x_0) \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact. The square on the right commutes by the naturality of α. It follows that $\exists$ unique dotted arrow α making the right-hand square commute. Composing with the map induced by the quotient map

$$X \times S^2/\{x_0\} \times S^2 \longrightarrow X \wedge S^2,$$

we obtain the desired

(1) $$\alpha : \tilde{K}(X \wedge S^2) \longrightarrow \tilde{K}(X).$$

Clearly α is natural in X, and by construction the square

$$\begin{tikzcd}
\tilde{K}(X \wedge S^2) \arrow[r, "\alpha"] \arrow[d] & \tilde{K}(X) \arrow[d] \\
K(X \times S^2) \arrow[r, "\alpha"] & K(X)
\end{tikzcd}$$

(where the vertical maps are induced by the quotient map $X \times S^2 \to X \wedge S^2$ and inclusions $\tilde{K} \hookrightarrow K$) commutes.

We will show that the map α in (1) is an inverse for the map $\beta : \tilde{K}(X) \longrightarrow \tilde{K}(X \wedge S^2)$.

Lemma 5: $\alpha\beta = \mathrm{id}_{\tilde{K}(X)}$.

Pf: We have the commutative diagram

$$\begin{tikzcd}
\tilde{K}(X) \arrow[r, "\beta"] \arrow[d, hook] & \tilde{K}(X\wedge S^2) \arrow[r, "\alpha"] \arrow[d] & \tilde{K}(X) \arrow[d, hook] \\
K(X) \arrow[r, "\beta"] & K(X\times S^2) \arrow[r, "\alpha"] & K(X)
\end{tikzcd}$$

where the middle vertical map is induced by the quotient map $X\times S^2 \to X\wedge S^2$. In the bottom row, we have $\beta(1) = \pi_{S^2}^*(b)$ ($\pi_{S^2}: X\times S^2 \to S^2$ the projection), and by naturality and property 3) in Prop. 4 we have $\alpha(\pi_{S^2}^*(b)) = 1$. Since both α and β in the bottom row are $K(X)$-linear, it follows that $\alpha\beta = \mathrm{id}_{K(X)}$ in the bottom row, which implies that $\alpha\beta = \mathrm{id}_{\tilde{K}(X)}$ in the top row. □

To show that $\beta\alpha = \mathrm{id}_{\tilde{K}(X\wedge S^2)}$, we consider the diagram

$$\begin{tikzcd}
\tilde{K}(X\wedge S^2) \arrow[rr, "\beta"] \arrow[dr] \arrow[ddd, "\alpha"'] & & \tilde{K}(X\wedge S^2\wedge S^2) \arrow[rr, "(1\wedge T_1)^*"] \arrow[d] & & \tilde{K}(X\wedge S^2\wedge S^2) \arrow[dl] \arrow[ddd, "\alpha"] \\
& K(X\times S^2) \arrow[r, "\beta"] \arrow[d, "\alpha"'] & K(X\times S^2\times S^2) \arrow[r, "(1\times T_2)^*"] \arrow[d, phantom, "(*)"] & K(X\times S^2\times S^2) \arrow[d, "\alpha"] & \\
& K(X) \arrow[rr, "\beta"] & & K(X\times S^2) & \\
\tilde{K}(X) \arrow[ur] \arrow[rrrr, "\beta"] & & & & \tilde{K}(X\wedge S^2) \arrow[ul, hook]
\end{tikzcd}$$

Here

- $T_1 : S^2 \wedge S^2 \longrightarrow S^2 \wedge S^2$ and $T_2 : S^2 \times S^2 \longrightarrow S^2 \times S^2$ are the homeomorphisms interchanging the two factors,
- unlabeled maps are induced by the evident quotient maps and inclusions $\tilde{K} \hookrightarrow K$.
- the rectangle (*) in the middle commutes by properties 1) and 2) in Prop. 4.
- it is easy to see that the rest of the diagram commutes as well.

Since the map $\tilde{K}(X \wedge S^2) \hookrightarrow K(X \times S^2)$ in the bottom right-hand corner is a monomorphism, the commutativity of the diagram implies that the outer rectangle in the diagram commutes.