Reduction to finding α

Recall that we are trying to prove

Thm 1 (Bott periodicity): Let X be a pointed compact Hausdorff space. Then the map

$$\tilde{K}(X) \xrightarrow{\beta} \tilde{K}(X \wedge S^2)$$

given by reduced external product with the Bott class $b = [H] - 1 \in \tilde{K}(S^2)$ is an isomorphism.

Our strategy is to reduce Thm 1 to

Prop 2: For every compact Hausdorff space X, there exists a map

$$\alpha : K(X \times S^2) \longrightarrow K(X)$$

s.t.

1) α is natural in X
2) α is $K(X)$-linear
3) For $X = pt$ and $b \in K(S^2)$ the Bott class, $\alpha(b) = 1 \in K(pt)$.

Assuming that such an α exists, we constructed a reduced version

$$\alpha : \tilde{K}(X \wedge S^2) \longrightarrow \tilde{K}(X)$$

which we claim to be an inverse for β. We saw

<u>Lemma 3</u>: The composite

$$\tilde{K}(X) \xrightarrow{\beta} \tilde{K}(X \wedge S^2) \xrightarrow{\alpha} \tilde{K}(X)$$

is the identity map. $\square$

We also showed that the diagram

$$\begin{tikzcd}
\tilde{K}(X \wedge S^2) \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \tilde{K}(X \wedge S^2 \wedge S^2) \arrow[r, "(id_X \wedge T)^*"] & \tilde{K}(X \wedge S^2 \wedge S^2) \arrow[d, "\alpha"] \\
\tilde{K}(X) \arrow[rr, "\beta"] & & \tilde{K}(X \wedge S^2)
\end{tikzcd}$$

commutes, where $T: S^2 \wedge S^2 \longrightarrow S^2 \wedge S^2$ is the map $T(x \wedge y) = y \wedge x$.

<u>Lemma 4</u>: The map $\tilde{K}(X \wedge S^2 \wedge S^2) \xrightarrow{(id_X \wedge T)^*} \tilde{K}(X \wedge S^2 \wedge S^2)$ is the identity.

<u>Pf</u>: Regard $S^2 \wedge S^2 \approx S^4$ as the one-point compactification of $\mathbb{R}^4$. Then T is induced by the linear map

$$\tau : \mathbb{R}^4 \longrightarrow \mathbb{R}^4, \quad (x_1, x_2, x_3, x_4) \longmapsto (x_3, x_4, x_1, x_2).$$

Since $\det(\tau) = 1$, we can connect τ to $id_{\mathbb{R}^4}$ by a path in $GL_4(\mathbb{R})$. This yields a pointed homotopy from T to $id_{S^2 \wedge S^2}$, and hence from $id_X \wedge T$ to $id_{X \wedge S^2 \wedge S^2}$. $\square$

We conclude that in the diagram

$$\beta\alpha \underset{\substack{\uparrow\\ \text{diagram}}}{=} \alpha(\mathrm{id}_X \wedge T)^*\beta \underset{\substack{\uparrow\\ \text{Lemma 4}}}{=} \alpha\beta \underset{\substack{\uparrow\\ \text{Lemma 3}}}{=} \mathrm{id}_{\tilde{K}(X\wedge S^2)} .$$

Conclusion: $\alpha : \tilde{K}(X\wedge S^2) \longrightarrow \tilde{K}(X)$ and $\beta : \tilde{K}(X) \to \tilde{K}(X\wedge S^2)$ are inverse isomorphisms, so Prop 2 implies Thm 1.

We are left with the task of constructing the maps α as in Prop. 2. We will do so by considering successively more general clutching functions.

Linear clutching functions

Let $\xi \to X$ be a complex v.b., and let $p : \pi^*\xi \to \pi^*\xi$ be a linear clutching function, i.e.

$$p(x,z) = a(x)z + b(x)$$

for some endomorphisms $a, b : \xi \to \xi$.
(As before, $\pi : X\times S^1 \longrightarrow X$ is the projection.)
We will use p to construct a decomposition

$$\xi = (\xi, p)_+ \oplus (\xi, p)_-$$

of ξ. To do so, we will construct a projection operator on ξ.

<u>Def 5</u>: A <u>projection</u> <u>operator</u> on a v.b. $\xi \to X$ is a v.b. endomorphism $Q : \xi \to \xi$ s.t. $Q^2 = Q$.

Write $Q_x : \xi_x \to \xi_x$ for the restriction of Q. Notice that for each $x \in X$, $\mathrm{Im}(Q_x) \cap \mathrm{Ker}(Q_x) = 0$, so ξ_x decomposes as $\xi_x = \mathrm{Im}(Q_x) \oplus \mathrm{Ker}(Q_x)$. We would like to promote this to a decomposition of vector bundles. For this, we need

<u>Prop 6</u>: Any projection operator Q on a v.b. $\xi \to X$ is of locally constant rank (i.e. $\mathrm{rank}(Q_x : \xi_x \to \xi_x)$ is a locally constant function of $x \in X$).

<u>Pf</u>: Let $x \in X$. Use a local trivialization of ξ near x to construct sections $s_1, \dots, s_n$ of ξ defined in a neighbourhood of x s.t. $s_1(x), \dots, s_r(x)$ are a basis of $\mathrm{Im}(Q_x)$ and $s_{r+1}(x), \dots, s_n(x)$ are a basis of $\mathrm{Ker}(Q_x)$. Then for y near x, the vectors

$$Qs_1(y), \dots, Qs_r(y), (1-Q)s_{r+1}(y), \dots, (1-Q)s_n(y) \in \xi_y$$

form a basis of ξ_y (since they do at $y = x$). Since $Qs_1(y), \dots, Qs_r(y) \in \mathrm{Im}\, Q_y$ and $(1-Q)s_{r+1}(y), \dots, (1-Q)s_n(y) \in \mathrm{Ker}\, Q_y$, it follows that $\dim(\mathrm{Im}\, Q_y) \geq r$ and $\dim(\mathrm{Ker}\, Q_y) \geq n-r$. Since $\dim(\mathrm{Im}\, Q_y) + \dim(\mathrm{Ker}\, Q_y) = \dim(\xi_y) = n$, it follows that

$$\mathrm{rank}\, Q_y = \dim(\mathrm{Im}\, Q_y) = r. \qquad \square$$

By the generalization of the construction of kernel and image of a constant-rank map of v.b.'s to the context of v.b.'s with varying fibre dimension, we obtain v.b.'s

$$\operatorname{Im} Q \to X \quad \text{and} \quad \operatorname{Ker} Q \to X$$

with fibres $(\operatorname{Im} Q)_x = \operatorname{Im}(Q_x)$ and $(\operatorname{Ker} Q)_x = \operatorname{Ker}(Q_x)$. Now

$$\xi = \operatorname{Im} Q \oplus \operatorname{Ker} Q$$

as vector bundles.

Let us now define a projection operator $Q_p : \xi \to \xi$ associated to the linear clutching function

$$p(x,z) = a(x)z + b(x)$$

as follows:

$$Q_p(x) = \frac{1}{2\pi i} \int_{|z|=1} [a(x)z + b(x)]^{-1} a(x)\, dz.$$

<u>Lemma 7</u>: $Q_p : \xi \to \xi$ is a projection operator.

<u>Pf</u>: First notice that for $z \neq w$, we have

$$(*)\left\{\begin{aligned} \frac{(az+b)^{-1}}{w-z} + \frac{(aw+b)^{-1}}{z-w} &= (az+b)^{-1}\frac{aw+b}{w-z}(aw+b)^{-1} \\ &\quad + (az+b)^{-1}\frac{az+b}{z-w}(aw+b)^{-1} \\ &= (az+b)^{-1}a(aw+b)^{-1}. \end{aligned}\right.$$

Since $a(x)z+b(x)$ is invertible for all $x\in X$, $|z|=1$, we can find an $\varepsilon>0$ s.t. it is invertible for all $x\in X$, $1-\varepsilon<|z|<1+\varepsilon$. Now the Cauchy integral theorem implies that

$$Q_p(x)=\frac{1}{2\pi i}\int_{|z|=r}[a(x)z+b(x)]^{-1}a(x)\,dz$$

for any $1-\varepsilon<r<1+\varepsilon$. Let $1-\varepsilon<r<R<1+\varepsilon$.

Then

$$Q_p^2=\frac{1}{(2\pi i)^2}\int_{|z|=R}(az+b)^{-1}a\,dz\int_{|w|=r}(aw+b)^{-1}a\,dw$$

linearity

$$=\frac{1}{(2\pi i)^2}\int_{|w|=r}\int_{|z|=R}(az+b)^{-1}a\,(aw+b)^{-1}a\,dz\,dw$$

By $(*)$

$$=\frac{1}{(2\pi i)^2}\int_{|w|=r}\int_{|z|=R}\frac{(az+b)^{-1}}{w-z}a\,dz\,dw+\frac{1}{(2\pi i)^2}\int_{|w|=r}\int_{|z|=R}\frac{(aw+b)^{-1}}{z-w}a\,dz\,dw$$

Fubini, Cauchy integral formula

$$=\frac{1}{(2\pi i)^2}\int_{|z|=R}\int_{|w|=r}\frac{(az+b)^{-1}}{w-z}a\,dw\,dz+\frac{1}{2\pi i}\int_{|w|=r}(aw+b)^{-1}a\,dw$$

Cauchy integral theorem

$$=\frac{1}{2\pi i}\int_{|z|=R}0\,dz+Q_p=Q_p.\qquad\square$$

Cor 8: The v.b. $\zeta \to X$ decomposes as a direct sum

$$\zeta = (\zeta,p)_+ \oplus (\zeta,p)_-$$

where $(\zeta,p)_+ = \operatorname{Im} Q_p$ and $(\zeta,p)_- = \operatorname{Ker} Q_p$.

Prop 9: We have

(i) $(\zeta, \overset{\mathrm{id}_\zeta}{1})_+ = \varepsilon^0_X$ and $(\zeta, \overset{z\,\mathrm{id}_\zeta}{z})_+ = \zeta$,

(ii) $(\zeta,p_0)_+ \approx (\zeta,p_1)_+$ if $p_0, p_1 : \pi^*\zeta \xrightarrow{\approx} \pi^*\zeta$ are homotopic through linear clutching functions.

(iii) $(\zeta_1 \oplus \zeta_2, p_1 \oplus p_2)_+ \approx (\zeta_1,p_1)_+ \oplus (\zeta_2,p_2)_+$

(iv) $(\xi \otimes \zeta, \mathrm{id}_\xi \otimes p)_+ \approx \xi \otimes (\zeta,p)_+$ for any complex v.b. $\xi \to X$.

(Here the ζ's are complex v.b.'s over X and p's are linear clutching functions.)

Pf: (i):

$$Q_1 = \frac{1}{2\pi i} \int_{|z|=1} \mathrm{id}_\zeta \, dz = 0$$ (by Cauchy integral thm)

$$Q_2 = \frac{1}{2\pi i} \int_{|z|=1} \frac{\mathrm{id}_\zeta}{z} \, dz = \mathrm{id}_\zeta$$ (by Cauchy integral formula or direct computation)

(ii): A homotopy from p_0 to p_1 through linear clutching functions gives a linear clutching function P for $\xi\times I \longrightarrow X \times I$, and $(\xi, p_0)_+$ and $(\xi, p_1)_+$ are the two ends of the v.b.

$$(\xi\times I, P)_+ \longrightarrow X\times I.$$

(iii) & (iv) are immediate from the construction. □