Polynomial clutching functions

Recall: We are trying to prove Bott periodicity, and we have reduced the task to constructing maps

$$\alpha : K(X \times S^2) \longrightarrow K(X)$$

s.t.

1) α is natural in X
2) α is $K(X)$-linear
3) $\alpha(b) = 1 \in K(pt)$, where $b = [H] - 1 \in K(S^2)$ is the Bott class.

Last time, given a complex v.b. $\xi \to X$ and a linear clutching function $p : \pi^*\xi \xrightarrow{\approx} \pi^*\xi$ ($\pi : X \times S^1 \to X$ the projection), we constructed a direct sum decomposition

$$\xi = (\xi, p)_+ \oplus (\xi, p)_-$$

of v.b.'s over X.

Suppose now $p(x,z) = \sum_{k=0}^{n} a_k(x) z^k : \pi^*\xi \to \pi^*\xi$ is a polynomial clutching function of degree $\leq n$. We set

$$\mathcal{L}^n(\xi) := \oplus^{n+1}\xi \longrightarrow X$$

and define a linear clutching function

$$\alpha^n(p) : \pi^*\alpha^n(\xi) \xrightarrow{\approx} \pi^*\alpha^n(\xi)$$

by

$$\alpha^n(p) = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n \\ -z & 1 & 0 & \cdots & 0 & 0 \\ 0 & -z & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -z & 1 \end{pmatrix} .$$

(Notice that then

$$\alpha^n(p) = \begin{pmatrix} 1 & q_1 & q_2 & \cdots & q_n \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} p & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ -z & 1 & & \\ & -z & \ddots & \\ & & \ddots & \\ & & -z & 1 \end{pmatrix}$$

where the polynomials q_i are defined inductively by $q_0 = p$, $q_{r+1}(z) = (q_r(z) - q_r(0))/z$; it follows that $\alpha^n(p)$ indeed is an isomorphism.)

We set

$$\alpha^n(\xi, p)_\pm := (\alpha^n(\xi), \alpha^n(p))_\pm \longrightarrow X .$$

<u>Prop 1</u>: We have

(i) $\alpha^{n+1}(\xi, p)_+ \approx \alpha^n(\xi, p)_+$ and

$\alpha^{n+1}(\xi, zp)_+ \approx \alpha^n(\xi, p)_+ \oplus \xi$

if p is polynomial of degree $\leq n$.

(ii) $\mathcal{L}^n(\xi, p_0)_+ \approx \mathcal{L}^n(\xi, p_1)_+$ if p_0 and p_1 are homotopic through polynomial clutching functions of degree $\leq n$.

(iii) $\mathcal{L}^n(\xi_1 \oplus \xi_2, p_1 \oplus p_2)_+ \approx \mathcal{L}^n(\xi_1, p_1)_+ \oplus \mathcal{L}^n(\xi_2, p_2)_+$ (p_1, p_2 polynomial of degree $\leq n$)

(iv) $\mathcal{L}^n(\eta \otimes \xi, \mathrm{id}_{\pi^*(\xi)} \otimes p)_+ \approx \eta \otimes \mathcal{L}^n(\xi, p)_+$ (p polynomial of degree $\leq n$, $\eta \to X$ a complex v.b.)

Pf: (i): Notice that

$$\left(\begin{array}{c|c} \mathcal{L}^n(p) & 0 \\ \hline 0 \;\cdots\; -tz & 1 \end{array}\right), \quad 0 \leq t \leq 1$$

gives a homotopy from $\mathcal{L}^n(p) \oplus 1$ to $\mathcal{L}^{n+1}(p)$ through linear clutching functions. On the other hand, for $\alpha : I \to GL_2(\mathbb{C})$ a path connecting I_2 to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the product

$$\left(\begin{array}{c|c} \alpha(t) & 0 \\ \hline 0 & I_n \end{array}\right) \begin{pmatrix} 0 & a_0 & a_1 & \cdots & a_n \\ -z & 1-t & & & \\ & -z & 1 & & \\ & & \ddots & \ddots & \\ & & & -z & 1 \end{pmatrix}, \quad 0 \leq t \leq 1$$

gives a homotopy from $\mathcal{L}^{n+1}(zp)$ to $z \oplus \mathcal{L}^n(p)$ through linear clutching functions. The claim now follows from Prop. XIX.9.(ii), (iii) and (i).

(ii): A homotopy from p_0 to p_1 through polynomial clutching functions yields a homotopy from $\mathcal{L}^n(p_0)$ to $\mathcal{L}^n(p_1)$ through linear clutching functions. The claim now follows from Prop. $\underline{XIX}$.9.(ii).

(iii) and (iv) are immediate from the construction and Prop. $\underline{XIX}$.9.(iii), (iv). $\square$

General clutching functions

Given a complex v.b. $\xi \longrightarrow X$ and a Laurent polynomial clutching function

$$\ell : \pi^*\xi \xrightarrow{\approx} \pi^*\xi,$$

for n large enough, $z^n \ell$ is a polynomial clutching function of degree $\leq 2n$. We define

$$\alpha(\xi, \ell) := n\xi - \mathcal{L}^{2n}(\xi, z^n\ell)_+ \in K(X).$$

By Prop. 1.(i), $\alpha(\xi,\ell)$ is independent of the choice of $n \gg 0$.

By Prop. 1.(ii), we have $\alpha(\xi, \ell_0) = \alpha(\xi, \ell_1)$ if ℓ_0 and ℓ_1 are homotopic through Laurent polynomial clutching functions. By Cor. $\underline{XVIII}$.3.(ii), it follows that more generally, $\alpha(\xi,\ell_0) = \alpha(\xi, \ell_1)$ whenever ℓ_0, ℓ_1 are Laurent polynomial clutching functions homotopic through clutching functions.

For an arbitrary clutching function

$$u : \pi^*\xi \xrightarrow{\approx} \pi^*\xi,$$

we now obtain a well-defined element

$$\alpha(\xi, u) \in K(X)$$

by setting $\alpha(\xi,u) = \alpha(\xi,\ell)$, where ℓ is a Laurent polynomial clutching function homotopic to u through clutching functions. (Such an ℓ exists by Cor. XVIII.3.(i).)

Prop 2: We have

(i) $\alpha(\xi, u_0) = \alpha(\xi, u_1) \in K(X)$ if u_0 and u_1 are homotopic through clutching functions $\pi^*\xi \xrightarrow{\approx} \pi^*\xi$.

(ii) $\alpha(\xi_1 \oplus \xi_2, u_1 \oplus u_2) = \alpha(\xi_1,u_1) + \alpha(\xi_2,u_2) \in K(X)$ ($\xi_1, \xi_2 \to X$ complex v.b.'s, $u_i : \pi^*\xi_i \xrightarrow{\approx} \pi^*\xi_i$, $i=1,2$, clutching functions).

(iii) $\alpha(\eta \otimes \xi, id_{\pi^*(\eta)} \otimes u) = \eta\, \alpha(\xi, u) \in K(X)$ ($\eta, \xi \to X$ complex v.b.'s, $u : \pi^*\xi \xrightarrow{\approx} \pi^*\xi$ clutching function).

Pf: (i) is immediate from the definition. (ii) and (iii) follow from Prop. 1.(iii) and (iv), respectively. □

Construction of $\alpha : K(X \times S^2) \longrightarrow K(X)$

For a complex v.b. $\xi \to X \times S^2$, set $\zeta = \xi|X$ (where we identify X with $X \times \{1\} \subset X \times S^2$), and define

$$\alpha(\xi) := \alpha(\zeta, u) \in K(X)$$

where $u : \pi^*\zeta \xrightarrow{\approx} \pi^*\zeta$ is the clutching function (unique up to homotopy) afforded by Prop. XVIII.4. (Then $\xi \approx \pi_0^*\zeta \cup_u \pi_\infty^*\zeta$.) By Prop. 2.(i), $\alpha(\xi)$ is well defined. Prop. 2.(ii) and (iii) imply

Prop 3: We have

(i) $\alpha(\xi_1 \oplus \xi_2) = \alpha(\xi_1) + \alpha(\xi_2) \in K(X)$
  ($\xi_1, \xi_2 \to X \times S^2$ complex v.b.'s).

(ii) $\alpha(\pi_X^*(\eta) \otimes \xi) = \eta\, \alpha(\xi)$
  ($\eta \to X$, $\xi \to X \times S^2$ complex v.b.'s, $\pi_X : X \times S^2 \to X$ the projection). $\square$

By Prop. 3.(i), the map

$$\mathrm{Vect}_{\mathbb{C}}(X \times S^2) \xrightarrow{\alpha} K(X)$$

induces a homomorphism

$$K(X \times S^2) \xrightarrow{\alpha} K(X)$$

of abelian groups, and by Prop. 3.(ii), the induced map is $K(X)$-linear. Tracing through the construction, we see that α is natural in X. It remains to show that $\alpha(b) = 1 \in K(pt)$ for $b = [H] - 1 \in K(S^2)$. This follows from

Lemma 4: (i) $\alpha(H) = 1 \in K(pt)$

(ii) $\alpha(\varepsilon^1_{S^2}) = 0 \in K(pt)$.

Pf: (i): From Lecture 11 it follows that as the clutching function $u : \pi^* H_1 \xrightarrow{\approx} \pi^* H_1$ we may pick $u = z^{-1}$. Thus

$$\alpha(H) = \alpha(H_1, z^{-1}) = H_1 - \mathcal{L}^2(H_1, 1)_+$$
$$= 1 - (\mathcal{L}^2(H_1), \mathcal{L}^2(1))_+ \in K(pt).$$

Here

$$\mathcal{L}^2(1) = \begin{pmatrix} 1 & 0 & 0 \\ -z & 1 & 0 \\ 0 & -z & 1 \end{pmatrix} \underset{\substack{\uparrow \\ \text{through linear} \\ \text{clutching functions}}}{\simeq} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus by Prop. XIX.9.(ii), (iii) and (i) we have

$$(\mathcal{L}^2(H_1), \mathcal{L}^2(1))_+ = 0 \in K(pt),$$

and the claim follows.

(ii): As the clutching function $u : \pi^* \varepsilon^1_{pt} \xrightarrow{\approx} \pi^* \varepsilon^1_{pt}$ we may pick the identity map 1. Thus

$$\alpha(\varepsilon^1_{S^2}) = \alpha(\varepsilon^1_{pt}, 1) = -\mathcal{L}^0(\varepsilon^1_{pt}, 1)_+$$
$$= -(\varepsilon^1_{pt}, 1)_+ \underset{\substack{\uparrow \\ \text{Prop. XIX.9.(i)}}}{=} \varepsilon^0_{pt} = 0 \in K(pt). \quad \square$$

This concludes the proof of Bott periodicity.