## Positive K-groups

Let $X$ be a pointed compact Hausdorff space. We have already defined $\tilde{K}^n(X)$ for $n \leq 0$, and by Bott periodicity, we have an isomorphism

$$\tilde{K}^n(X) \xrightarrow[\approx]{\beta} \tilde{K}^{n-2}(X)$$

for all $n \leq 0$. In view of this, we extend the definition of $\tilde{K}^n(X)$ to positive $n$ as follows:

Def 1: For $n > 0$, we set

$$\tilde{K}^n(X) := \begin{cases} \tilde{K}^0(X) & \text{if } n \text{ is even} \\ \tilde{K}^{-1}(X) & \text{if } n \text{ is odd} \end{cases} .$$

Then $\tilde{K}^n(X) \approx \tilde{K}^{n-2}(X)$ for all $n \in \mathbb{Z}$.
As before, we obtain unreduced K-groups $K^n(X)$ by substituting $X_+$ for $X$ and relative K-groups $K^n(X, A)$ by substituting $X/A$ for $X$. Induced maps between the higher K-groups are defined in the obvious way.

We obtain a $\mathbb{Z}$-graded abelian group

$$\tilde{K}^*(X) = \bigoplus_{n \in \mathbb{Z}} \tilde{K}^n(X) .$$

In view of the periodicity, however, we

usually think of $\tilde{K}^*(X)$ as $\mathbb{Z}/2$-graded instead:

$$\boxed{\tilde{K}^*(X) = \tilde{K}^0(X) \oplus \tilde{K}^1(X).}$$

Similarly for $K^*(X)$ and $K^*(X,A)$.

The reduced $K$-groups have the following basic properties ($X, Y$ pointed compact Hausdorff spaces, $n \in \mathbb{Z}$):

1) Homotopy invariance: the induced map
$$\tilde{K}^n(Y) \xrightarrow{f^*} \tilde{K}^n(X)$$
only depends on the pointed homotopy class of $f: X \longrightarrow Y$.

2) Suspension: for all $n$, there is a natural isomorphism
$$\tilde{K}^n(X) \approx \tilde{K}^{n+1}(\Sigma X).$$

3) Exactness: for $A \subset X$ a closed subspace containing the basepoint, the sequence
$$\tilde{K}^n(X/A) \xrightarrow{q^*} \tilde{K}^n(X) \xrightarrow{i^*} \tilde{K}^n(A)$$
is exact. (Here $i: A \hookrightarrow X$ is the inclusion, $q: X \to X/A$ is the quotient map.)

4) Additivity: the inclusions $i_X: X \hookrightarrow X \vee Y$, $i_Y: Y \hookrightarrow X \vee Y$ induce an isomorphism
$$\tilde{K}^n(X \vee Y) \xrightarrow[\approx]{(i_X^*, i_Y^*)} \tilde{K}^n(X) \times \tilde{K}^n(Y).$$

These are similar to properties enjoyed by reduced ordinary cohomology groups, so $\tilde{K}^*$ is called a (reduced) <u>generalized</u> <u>cohomology</u> <u>theory</u>.

Sketch of proof of the properties:

1) follows from the homotopy invariance of induced maps on $\tilde{K}^0$ (lecture 13), which in turn follows from the homotopy invariance of induced maps on $K^0$ (Prop. XII.7). One needs the observation that $\Sigma f$ and $\Sigma g$ are pointed homotopic if $f$ and $g$ are.

2) follows immediately from the definitions and Bott periodicity.

3) follows immediately from the exactness on nonpositive K-groups (lecture 16).

4) follows from the case $n=0$, which is Cor XV.3, and the natural homeomorphism $\Sigma(X \vee Y) \approx \Sigma X \vee \Sigma Y$.

The proof of Cor. XV.3 was left as an exercise. Here is a proof. Let $q_X : X \vee Y \to X$, $q_Y : X \vee Y \to Y$ be the quotient maps. By Prop. XV.2, the sequence

$$\tilde{K}(Y) \xrightarrow{q_Y^*} \tilde{K}(X \vee Y) \xrightarrow{i_X^*} \tilde{K}(X)$$

is exact. The map $i_Y^*$ provides a retraction for $q_Y^*$ and the map $q_X^*$ a section for $i_X^*$.

Thus we have a short exact sequence

$$0 \to \tilde{K}(Y) \xrightarrow{q_Y^*} \tilde{K}(X \vee Y) \xrightarrow{i_X^*} \tilde{K}(X) \to 0$$

with the map $i_Y^*$ providing a retraction for $q_Y^*$. The claim now follows from the proof of Lemma $\underline{XIII}$.6.(iii). □

It is a consequence of properties 1), 2) and 3) that there exists a long exact sequence

$$\cdots \xleftarrow{\delta} \tilde{K}^n(A) \xleftarrow{i^*} \tilde{K}^n(X) \xleftarrow{q^*} \tilde{K}^n(X/A) \xleftarrow{\delta} \tilde{K}^{n-1}(A) \xleftarrow{i^*} \cdots$$

Rather than deriving the sequence from 1), 2) and 3), however, it is quicker for us to construct it by using Bott periodicity to extend the long exact sequence for nonpositive K-groups as follows:

$$\begin{tikzcd}
\tilde{K}^0(A) & \tilde{K}^0(X) \arrow[l, "i^*"'] & \cdots \arrow[l, "q^*"'] & \tilde{K}^{-2}(A) \arrow[l, "\delta"'] & \tilde{K}^{-2}(X) \arrow[l, "i^*"'] \\
 & & & \tilde{K}^0(A) \arrow[u, "\beta", "\approx"'] & \tilde{K}^0(X) \arrow[l, "i^*"'] \arrow[u, "\beta", "\approx"'] & \cdots \arrow[l, "q^*"']
\end{tikzcd}$$

In view of the periodicity, the result is conveniently described as a six-term exact sequence

$$\begin{tikzcd}
\tilde{K}^0(X/A) \arrow[r, "q^*"] & \tilde{K}^0(X) \arrow[r, "i^*"] & \tilde{K}^0(A) \arrow[d] \\
\tilde{K}^1(A) \arrow[u] & \tilde{K}^1(X) \arrow[l, "i^*"'] & \tilde{K}^1(X/A) \arrow[l, "q^*"']
\end{tikzcd}$$

Substituting $X_+$ for $X$ and $A_+$ for $A$, this becomes

$$\begin{tikzcd}
K^0(X,A) \arrow[r, "j^*"] & K^0(X) \arrow[r, "i^*"] & K^0(A) \arrow[d] \\
K^1(A) \arrow[u] & K^1(X) \arrow[l, "i^*"'] & K^1(X,A) \arrow[l, "j^*"']
\end{tikzcd}$$

where $i : A \hookrightarrow X$ and $j : X = (X, \phi) \hookrightarrow (X, A)$ are the inclusions.

Unreduced K-groups have the following basic properties: ($(X,A), (Y,B)$ compact pairs, $n \in \mathbb{Z}$)

1) Homotopy invariance: the induced map

$$K^n(Y,B) \xrightarrow{f^*} K^n(X,A)$$

only depends on the homotopy class of $f : (X,A) \longrightarrow (Y,B)$ as a map of pairs.

2) Excision: if $U \subset X$ is an open subset s.t. $\bar{U} \subset A$, then the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism

$$K^n(X,A) \xrightarrow{\approx} K^n(X\setminus U, A \setminus U).$$

3) Exactness: there are natural homomorphisms $\delta : K^n(A) \longrightarrow K^{n+1}(X,A)$, $n \in \mathbb{Z}$, s.t. the sequence

$$\cdots \xrightarrow{\delta} K^n(X,A) \xrightarrow{j^*} K^n(X) \xrightarrow{i^*} K^n(A) \xrightarrow{\delta} K^{n+1}(X,A) \xrightarrow{j^*} \cdots$$

is exact. Here $i : A \hookrightarrow X$ and $j : X = (X, \phi) \hookrightarrow (X,A)$ are the inclusions.

4) Additivity: the inclusions

$$(X, A) \xrightarrow{i_X} (X \amalg Y, A \amalg B)$$

$$(Y, B) \xrightarrow{i_Y} (X \amalg Y, A \amalg B)$$

induce an isomorphism

$$K^n(X \amalg Y, A \amalg B) \xrightarrow{(i_X^*, i_Y^*)} K^n(X, A) \times K^n(Y, B)$$

Again, these are similar to properties enjoyed by ordinary cohomology (the Eilenberg-Steenrod axioms, with the dimension axiom omitted). One calls K-theory a generalized cohomology theory.

Sketch of proof of the properties:

1) follows immediately from the homotopy invariance for reduced K-theory.

2) is immediate from the homeomorphism

$$X \setminus U / A \setminus U \approx X/A.$$

3) follows from the long exact sequence on p. 4 by substituting $X_+$ for $X$ and $A_+$ for $A$.

4) follows from additivity for reduced K-theory. □

<u>First computations using Bott periodicity</u>

<u>Prop 2:</u> Let $n \geq 1$. As an abelian group, $\tilde{K}(S^{2n}) \approx \mathbb{Z}$ with generator $(H-1)^{*n}$. The product in $\tilde{K}(S^{2n})$ is trivial.

Pf: By repeated application of the periodicity theorem, we see that the iterated external product

$$\underbrace{\tilde{K}(S^2)\otimes \cdots \otimes \tilde{K}(S^2)}_{n} \xrightarrow{*} \tilde{K}(S^{2n})$$

is an isomorphism. As the external product preserves products (i.e. $(a*b)(a'*b') = (aa')*(bb')$), this is an isomorphism of rings (without unit). (Notice that it is immediate from the definition that the unreduced external product preserves products, and since the reduced external product is a restriction of the unreduced one, it also does.) The claim follows since $\tilde{K}(S^2) \approx \mathbb{Z}$ with generator $H-1$ and $(H-1)^2 = 0 \in \tilde{K}(S^2)$ by the relation $H \oplus H \approx H^{\otimes 2} \oplus \varepsilon^1$ proved in lecture 11. □

Cor 3: $K(S^{2n}) \approx \mathbb{Z}[x]/(x^2)$ as rings, where $x$ corresponds to the element $(H-1)^{*n} \in \tilde{K}(S^{2n}) \subset K(S^{2n})$. □

Lemma 4: $K(S^1) \approx \mathbb{Z}$.

Pf: Computing $K(S^1)$ was problem 8.4. Here is an argument: For all $n$, $\mathrm{Vect}^n_{\mathbb{C}}(S^1) \approx [S^0; GL_n\mathbb{C}] = pt$ since $GL_n\mathbb{C}$ is path connected. It follows that all complex v.b.'s over $S^1$ are trivial. Thus

$$(\mathrm{Vect}_{\mathbb{C}}(S^1), \oplus) \approx (\mathbb{N}, +),$$

and the claim follows. □

It follows that $\tilde{K}(S^1) = 0$. Combining this with Bott periodicity, we get

Prop 5: For all $n \geq 0$, $\tilde{K}(S^{2n+1}) = 0$ and $K(S^{2n+1}) = \mathbb{Z}$. □

From the suspension isomorphism

$$\tilde{K}^n(X) \approx \tilde{K}^{n+1}(\Sigma X)$$

and Propositions 2 and 5 we get

Prop 6: $\tilde{K}^1(S^{2n}) = 0$ and $\tilde{K}^1(S^{2n+1}) \approx \mathbb{Z}$. □

Thus $\tilde{K}^*(S^n) \approx \mathbb{Z}$ for all $n \geq 0$, with the copy of $\mathbb{Z}$ occurring in degree 0 for $n$ even and in degree 1 for $n$ odd. From this computation, one can easily deduce the following deep result:

Thm 7 (Brouwer fixed point theorem): Any continuous map $f: D^n \to D^n$ has a fixed point: there exists $x \in D^n$ s.t. $f(x) = x$.

Pf: Problems 12.1-2. □

Prop. 2 and Bott periodicity imply

Prop 8: For any pointed compact Hausdorff space $X$, the external product

$$\tilde{K}(X) \otimes \tilde{K}(S^{2n}) \xrightarrow{*} \tilde{K}(X \wedge S^{2n})$$

is an isomorphism. □

Together with Lemma XVI.8, this implies

**Prop 9:** For any compact Hausdorff space $X$, the external product

$$K(X) \otimes K(S^{2n}) \xrightarrow{*} K(X \times S^{2n})$$

is an isomorphism.

**Def 10:** If $f : S^{n-1} \to A$, write $A \cup_f D^n$ for the quotient space $A \amalg D^n/\sim$, where $\sim$ identifies $x$ with $f(x)$ for all $x \in S^{n-1}$. Notice that then $A$ embeds into $A \cup_f D^n$ as a closed subspace. For $n = 0$, we interpret $S^{-1} = \emptyset$, $D^0 = \{0\}$, so then $A \cup_f D^0$ is just $A \amalg \{0\}$. We say that $X$ is obtained from a subspace $A \subset X$ by attaching an $n$-cell if $\exists f : S^{n-1} \to A$ and a homeomorphism $X \approx A \cup_f D^n$ which is the identity on $A$. Notice that then $X/A \approx S^n$. We call $X$ a finite cell complex if $\exists$ filtration $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_k = X$ s.t. for all $0 \le i \le k$, $X_i$ is obtained from $X_{i-1}$ by attaching an $n_i$-cell for some $n_i \ge 0$.

**Example:** Recall that

$$\mathbb{C}P^n = \mathbb{C}^{n+1} \setminus \{0\} / \sim$$

where $\sim$ identifies $(z_0, \ldots, z_n)$ with $\lambda(z_0, \ldots, z_n)$

for all $\lambda \in \mathbb{C} \setminus \{0\}$. We write $[z_0 : \cdots : z_n]$ for the equivalence class of $(z_0, \ldots, z_n)$. The inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$, $v \mapsto (v, 0)$ induces an embedding $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$, $[v] \mapsto [v : 0]$. The space $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a $2n$-cell along the composite map

$$q : S^{2n-1} \hookrightarrow \mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{C}P^{n-1} :$$

the required homeomorphism

$$\mathbb{C}P^{n-1} \cup_q D^{2n} \approx \mathbb{C}P^n$$

is induced by the inclusion $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ and the map

$$D^{2n} \longrightarrow \mathbb{C}P^n, \quad v \longmapsto [v : 1 - \|v\|] .$$

Thus $\mathbb{C}P^n$ is a finite cell complex with $n+1$ cells, one of dimension $2k$ for each of $0 \le k \le n$.

<u>Example</u>: More generally, any finite CW complex is a finite cell complex. The difference is that in a cell complex, we do not require that cells are only attached to cells of lower dimension.

<u>Prop 11</u>: Let $X$ be a finite cell complex with $n$ cells. Then $K^*(X)$ is a finitely generated abelian group with $\leq n$ generators. If all cells have even dimension, then $K^1(X) = 0$ and $K^0(X)$ is a free abelian group of rank $n$.

<u>Pf</u>: Next time. $\square$