First computations using Bott periodicity (cont.)

Prop 1: Let X be a finite cell complex with n cells. Then $K^*(X)$ is a finitely generated abelian group with $\leq n$ generators. If all cells have even dimension, then $K^1(X) = 0$ and $K^0(X)$ is free abelian of rank n.

Pf: By induction on the number of cells. If $n=0$, then $X = \emptyset$ and $K^*(X) = 0$, so the claim holds. Suppose X is obtained from a subcomplex A by attaching ~~an~~ a k-cell. In the exact sequence

$$K^*(X,A) \longrightarrow K^*(X) \longrightarrow K^*(A)$$

we have $K^*(X,A) = \tilde{K}^*(X/A) \approx \tilde{K}^*(S^k) \approx \mathbb{Z}$, so $K^*(X)$ requires at most one more generator than $K^*(A)$. If all cells are even-dimensional, then the first term in the exact sequence

$$K^1(X,A) \longrightarrow K^1(X) \longrightarrow K^1(A)$$

is zero, so $K^1(X) = 0$ by induction on the number of cells. It follows that we have a short exact sequence

$$0 \to \underset{\mathbb{Z}}{\underset{\wr\wr}{K^0(X,A)}} \to K^0(X) \longrightarrow K^0(A) \to 0 .$$

By induction, $K^0(A)$ is free, so the sequence splits and we have $K^0(X) \approx \mathbb{Z} \oplus K^0(A)$. $\square$

<u>Cor 2</u>: $K^1(\mathbb{C}P^n) = 0$ and $K^0(\mathbb{C}P^n) \underset{\substack{\uparrow \\ \text{as abelian} \\ \text{groups}}}{\approx} \mathbb{Z}^{n+1}$. $\square$

The following result also follows from Prop 1, although it can more simply be read off from our computation of $\tilde{K}^*(S^n)$.

<u>Prop 3</u>: $K^0(pt) \approx \mathbb{Z}$ and $K^1(pt) = 0$. $\square$

<u>Prop 4</u>: Let X be a pointed compact Hausdorff space, let $i : pt \to X$ be the map onto the basepoint, and let $p : X_+ \to X$ be the quotient map which is the identity on X and sends the additional point to the basepoint. Then the sequence

$$0 \to \tilde{K}^*(X) \xrightarrow{p^*} K^*(X) \xrightarrow{i^*} K^*(pt) \to 0$$

is exact and split by the map $r^* : K^*(pt) \to K^*(X)$ induced by the unique map $r : X \to pt$.

<u>Pf</u>: The sequence

$$pt_+ \xrightarrow{i_+} X_+ \xrightarrow{p} X$$

gives us a long exact sequence

$$\cdots \xrightarrow{\delta} \tilde{K}^n(X) \xrightarrow{p^*} \tilde{K}^n(X_+) \xrightarrow{i_+^*} \tilde{K}^n(pt_+) \xrightarrow{\delta} \tilde{K}^{n+1}(X) \xrightarrow{p^*} \cdots$$

The map $r_+^* : \tilde{K}^n(pt_+) \to \tilde{K}^n(X_+)$ gives a section for i_+^*, so the above long exact sequence reduces to short exact sequences

$$0 \to \tilde{K}^n(X) \xrightarrow{p^*} \tilde{K}^n(X_+) \xrightarrow{i_+^*} \tilde{K}^n(pt_+) \to 0, \quad n \in \mathbb{Z},$$

each split by r_+^*. $\square$

Cor 5: $p^* : \tilde{K}^1(X) \xrightarrow{\approx} K^1(X)$ for all X. $\square$

Cor 6: $K^*(X) \approx \tilde{K}^*(X) \oplus K^*(pt)$. $\square$

Rk: In degree 0, Prop 4 essentially just recovers our definition of $\tilde{K}^0$: recall that we defined

$$\tilde{K}(X) = \operatorname{Ker}\left(K(X) \xrightarrow{i^*} K(pt)\right).$$

More about (external) products

The external product

$$\tilde{K}^{\leq 0}(X) \otimes \tilde{K}^{\leq 0}(Y) \xrightarrow{*} \tilde{K}^{\leq 0}(X \wedge Y)$$

extends by periodicity to an external product

$$\tilde{K}^*(X) \otimes \tilde{K}^*(Y) \xrightarrow{*} \tilde{K}^*(X \wedge Y)$$

natural in X and Y. By substituting X_+ for X and Y_+ for Y, we get a product

$$K^*(X) \otimes K^*(Y) \xrightarrow{*} K^*(X \times Y).$$

More generally, by substituting X/A for X and Y/B for Y, we get a product

$$K^*(X, A) \otimes K^*(Y, B) \xrightarrow{*} K^*(X \times Y, X \times B \cup A \times Y).$$

The composite

$$\tilde{K}^*(X) \otimes \tilde{K}^*(X) \xrightarrow{*} \tilde{K}^*(X \wedge X) \xrightarrow{\Delta^*} \tilde{K}^*(X)$$

($\Delta : X \to X \wedge X$ the diagonal map $x \mapsto x \wedge x$) makes $\tilde{K}^*(X)$ into a graded ring (without unit) extending the ring structure on $\tilde{K}(X)$. One can show that the ring structure on $\tilde{K}^*(X)$ is graded commutative, i.e. $xy = (-1)^{\deg x \deg y} yx$ for homogeneous elements $x, y \in \tilde{K}^*(X)$. We might get back to this point if time allows or if turns out to be important for us.

By substituting X_+ for X, we obtain a product

$$K^*(X) \otimes K^*(X) \longrightarrow K^*(X)$$

making $K^*(X)$ into a unital graded ring extending the ring $K(X)$. More generally, for closed subspaces $A, B \subset X$, we have a product

$$K^*(X, A) \otimes K^*(X, B) \longrightarrow K^*(X, A \cup B)$$

defined as the composite

$$K^*(X, A) \otimes K^*(X, B) \xrightarrow{*} K^*(X \times X, X \times B \cup A \times X)$$

$$\xrightarrow{\Delta^*} K^*(X, A \cup B),$$

where $\Delta : (X, A \cup B) \longrightarrow (X \times X, X \times B \cup A \times X)$ is the diagonal map.

Division algebras and parallelizability of spheres

Our next major goal is to use K-theory to answer the question of when $\mathbb{R}^n$ admits the structure of a real division algebra and the related question of when the sphere S^{n-1} is parallelizable. My plan is to follow the treatment in Hatcher's draft of a book.

Def 7: A division algebra structure on $\mathbb{R}^n$ is a bilinear map $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t. $ab = 0$ implies $a = 0$ or $b = 0$. The structure is unital if $\exists e \in \mathbb{R}^n$ s.t. $ex = xe = x$ for all $x \in \mathbb{R}^n$.

Note: In a division algebra, the maps $x \mapsto ax$ and $x \mapsto xa$ are invertible for all $a \neq 0$, so in a unital division algebra every nonzero element has a left inverse and a right inverse.

Example: $\mathbb{R}$, $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$ show that $\mathbb{R}^n$ admits the structure of a division algebra at least when $n = 1, 2, 4$ or 8.

Lemma 8: If $\mathbb{R}^n$ admits the structure of a division algebra, it admits the structure of a unital division algebra.

Pf: Let $\Phi: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a division algebra structure. Pick $e \in \mathbb{R}^n \setminus \{0\}$. By composing Φ with a linear isomorphism $\mathbb{R}^n \to \mathbb{R}^n$ taking $\Phi(e,e)$ to e, we may assume that $\Phi(e,e) = e$. Let $\alpha = \Phi(-,e)$ and $\beta = \Phi(e,-)$. Now the composite

$$\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\alpha^{-1}\times\beta^{-1}} \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\Phi} \mathbb{R}^n$$

is a unital division algebra structure. $\square$

Def 9: A smooth manifold M is parallelizable if the tangent bundle TM of M is trivial.

Example: S^n is parallelizable at least when $n = 0, 1, 3$ or 7. In these cases, we may interpret S^n as the elements of unit length in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, respectively, and a trivialization of TS^n is given by the map

$$\begin{aligned} S^n \times T_1 S^n &\xrightarrow{\approx} TS^n \\ (x, v) &\longmapsto (x, xv) \end{aligned}$$

Here $1 \in S^n$ is the unit for the division algebra structure.

Our goal is to show

Thm 10:

(i) If $\mathbb{R}^n$ admits the structure of a division algebra, then $n = 1, 2, 4$ or 8.

(ii) If S^n is parallelizable, then $n = 0, 1, 3$ or 7.

The proof will take several lectures.

Rk: Even though our proof ~~is~~ will be based on K-theory, notice that Thm 10.(i) is a theorem in algebra. The statement of the result does not involve any topology.