Division algebras and parallelizability of spheres (cont.)

Recall that we want to show

Thm 1:

(i) If $\mathbb{R}^n$ admits the structure of a division algebra, then $n = 1, 2, 4$ or 8.

(ii) If S^n is parallelizable, then $n = 0, 1, 3$ or 7.

H-spaces and the Hopf invariant

Def 2: An H-space is a space X equipped with a continuous multiplication $\mu : X \times X \to X$ which has a two-sided identity element $e \in X$.

Note: There is some variation in the definition of an H-space in the literature: often, it is only assumed that $\mu(e,-)$ and $\mu(-,e)$ are homotopic to id_X instead of being equal to it. Under mild conditions on X and e, the two definitions are equivalent (in the sense that a μ satisfying the laxer definition can be homotoped to one satisfying the stricter one).

Lemma 3: If $\mathbb{R}^n$ is a division algebra or S^{n-1} is parallelizable, then S^{n-1} is an H-space.

Pf: Sp. $\mathbb{R}^n$ is a division algebra. By Lemma XXII.8, we may assume that it is unital.

Now
$$S^{n-1}\times S^{n-1} \longrightarrow S^{n-1}$$
$$(x,y) \longmapsto xy/|xy|$$
is an H-space structure on S^{n-1}.

So now S^{n-1} is parallelizable. Then $\exists$ linearly independent sections
$$s_1,\dots,s_{n-1} : S^{n-1} \longrightarrow TS^{n-1}.$$
The vectors $s_1(e_1),\dots,s_{n-1}(e_n)$ form a basis of $T_{e_1}S^{n-1} = \mathrm{span}(e_2,\dots,e_n) \subset \mathbb{R}^n$. Replacing $s_1,\dots,s_{n-1}$ by suitable linear combinations, we may assume that
$$s_1(e_1)=e_2, \dots, s_{n-1}(e_1)=e_n.$$
Using the Gram-Schmidt process, we may further assume that the vectors
$$x, s_1(x), \dots, s_{n-1}(x)$$
are orthonormal for all $x\in S^{n-1}$; notice that the Gram-Schmidt process does not change $s_1(e_1),\dots,s_{n-1}(e_1)$, since the vectors $e_1, s_1(e_1),\dots, s_{n-1}(e_1)$ are already orthonormal. For $x\in S^{n-1}$, let $\alpha_x \in O(n)$ be the element mapping the standard basis to $x, s_1(x),\dots, s_{n-1}(x)$. Now $(x,y)\mapsto \alpha_x(y)$ is an H-space structure on S^{n-1} with identity element e_1: we have $\alpha_{e_1} = \mathrm{id}$ and $\alpha_x(e_1)=x$ for all $x\in S^{n-1}$. $\square$

By Lemma 3, to prove Thm 1, it is enough to show that if S^{n-1} is an H-space, then $n=1,2,4$ or 8. The first step is to show that even-dimensional spheres are not H-spaces (except for S^0):

Prop 4: S^{2n} is not an H-space for any $n>0$.

Pf: Sp. $\mu: S^{2n}\times S^{2n} \longrightarrow S^{2n}$ is an H-space multiplication with identity element $e\in S^{2n}$. Let

$$i_1: S^{2n}\longrightarrow S^{2n}\times S^{2n},\quad i_1(x)=(x,e)$$
$$i_2: S^{2n}\longrightarrow S^{2n}\times S^{2n},\quad i_2(x)=(e,x)$$
$$i: pt \longrightarrow S^{2n},\quad pt\longmapsto e.$$

Then the following diagram commutes:

$$\begin{tikzcd}
 & K(S^{2n}) & K(S^{2n})\otimes K(pt) \arrow[l, "*"', "\approx"] \\
K(S^{2n}) \arrow[ur, "id"] \arrow[r, "\mu^*"] \arrow[dr, "id"'] & K(S^{2n}\times S^{2n}) \arrow[u, "i_1^*"] \arrow[d, "i_2^*"] & K(S^{2n})\otimes K(S^{2n}) \arrow[l, "*"', "\approx"] \arrow[u, "1\otimes i^*"'] \arrow[d, "i^*\otimes 1"] \\
 & K(S^{2n}) & K(pt)\otimes K(S^{2n}) \arrow[l, "*"', "\approx"]
\end{tikzcd}$$

The horizontal maps on the right are isomorphisms by Prop. $\boxed{XXI}$.9.

In view of the computation $K(S^{2n}) = \mathbb{Z}[x]/(x^2)$ (Cor $\boxed{XXI}$.3) and the observation that $i^*(x)=0$ (since $x\in \tilde{K}(S^{2n})$), the part on the left is

$$\begin{tikzcd}
 & \mathbb{Z}[\alpha]/(\alpha^2) \\
\mathbb{Z}[\gamma]/(\gamma^2) \arrow[ur, "\gamma \mapsto \alpha"] \arrow[r, "\mu^*"] \arrow[dr, "\gamma \mapsto \beta"'] & \mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2) \arrow[u, "i_1^*"', "\alpha \mapsto \alpha,\ \beta \mapsto 0"] \arrow[d, "i_2^*", "\alpha \mapsto 0,\ \beta \mapsto \beta"'] \\
 & \mathbb{Z}[\beta]/(\beta^2)
\end{tikzcd}$$

It follows that $\mu^*(\gamma) = \alpha + \beta + m\alpha\beta$ for some $m \in \mathbb{Z}$. Therefore

$$\mu^*(\gamma^2) = \mu^*(\gamma)^2 = (\alpha + \beta + m\alpha\beta)^2 = 2\alpha\beta \neq 0,$$

a contradiction since $\gamma^2 = 0$. $\square$

For a map $f : S^{4n-1} \longrightarrow S^{2n}$, write C_f for the space $S^{2n} \cup_f D^{4n}$. Since $\tilde{K}^1(S^{2n}) = \tilde{K}^1(S^{4n}) = 0$, the long exact sequence for

$$S^{2n} \xrightarrow{i} C_f \xrightarrow{q} S^{4n}$$

gives a short exact sequence

$$0 \longrightarrow \underset{\substack{\cap\\ \mathbb{Z}}}{\tilde{K}(S^{4n})} \xrightarrow{q^*} \tilde{K}(C_f) \xrightarrow{i^*} \underset{\substack{\cap\\ \mathbb{Z}}}{\tilde{K}(S^{2n})} \longrightarrow 0$$

Let $\alpha = q^*((H-1)^{*2n}) \in \tilde{K}(C_f)$

and let $\beta \in \tilde{K}(C_f)$ be s.t. $i^*(\beta) = (H-1)^{*n} \in \tilde{K}(S^{2n})$.

Since $i^*(\beta^2) = (i^*\beta)^2 = 0 \in \tilde{K}(S^{2n})$, we have

$$\beta^2 = h\alpha$$

for some $h \in \mathbb{Z}$.

Def 5: The integer h is the Hopf invariant of the map $f: S^{4n-1} \longrightarrow S^{2n}$.

Lemma 6: The integer h is independent of the choice of β.

Pf: Any other choice of β is of the form $\beta + m\alpha$ for some $m \in \mathbb{Z}$. We have

$$(\beta + m\alpha)^2 = \beta^2 + 2m\alpha\beta + m^2\alpha^2 = \beta^2 + 2m\alpha\beta,$$

so it is enough to show that $\alpha\beta = 0$. Since $i^*(\alpha\beta) = i^*(\alpha)i^*(\beta) = 0$, we have $\alpha\beta = k\alpha$ for some $k \in \mathbb{Z}$. Now

$$k^2\alpha = k\alpha\beta = \alpha\beta^2 = h\alpha^2 = 0.$$

Since α generates an infinite cyclic group, it follows that $k^2 = 0$, so $k = 0$ and $\alpha\beta = k\alpha = 0$. □

Lemma 7: If S^{2n-1} admits the structure of an H-space, then $\exists$ map $f: S^{4n-1} \longrightarrow S^{2n}$ of Hopf invariant ± 1.

Pf: Let $\mu: S^{2n-1} \times S^{2n-1} \longrightarrow S^{2n-1}$ be an H-space multiplication with neutral element $e \in S^{2n-1}$.

View S^{2n} as a union of two disks along the equator,

$$S^{2n} = D_+^{2n} \cup D_-^{2n},$$

and S^{4n-1} as

$$S^{4n-1} = \partial(D^{2n} \times D^{2n}) = \partial D^{2n} \times D^{2n} \cup D^{2n} \times \partial D^{2n}.$$

The maps

$$\begin{aligned} \partial D^{2n} \times D^{2n} &\longrightarrow D_+^{2n} \\ (x, y) &\longmapsto |y|\,\mu(x, y/|y|) \end{aligned}$$

$$\begin{aligned} D^{2n} \times \partial D^{2n} &\longrightarrow D_-^{2n} \\ (x, y) &\longmapsto |x|\,\mu(x/|x|, y) \end{aligned}$$

make sense and are continuous even for $x = 0$ and $y = 0$, and they fit together to define a map

$$f : S^{4n-1} \longrightarrow S^{2n}.$$

Let $\Phi : (D^{4n}, S^{4n-1}) = (D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n})) \longrightarrow (C_f, S^{2n})$ be the evident map $D^{4n} \longrightarrow C_f = S^{2n} \cup_f D^{4n}$. We have a commutative diagram

$$\beta \longmapsto \text{gen.}$$

$$\begin{tikzcd}
\tilde{K}(C_f) \arrow[r, "(\mathrm{incl})^*"] & \tilde{K}(S^{2n}) \\
K(C_f, D_+^{2n}) \arrow[u, "\approx"] \arrow[r, "(\mathrm{incl})^*"] \arrow[d, "\Phi^*"'] & K(S^{2n}, D_+^{2n}) \arrow[u, "\approx"'] \arrow[dd, "(\mathrm{incl})^*"', "\approx"] \\
K((D^{2n}, \partial D^{2n}) \times D^{2n}) \arrow[d, "(\mathrm{incl})^*"', "\approx"] & \\
K((D^{2n}, \partial D^{2n}) \times \{e\}) & K(D_-^{2n}, \partial D^{2n}) \arrow[l, "(f|)^*"', "\approx"] \\
\tilde{K}(S^{2n}) \arrow[u, "(\mathrm{quot})^*", "\approx"'] &
\end{tikzcd}$$

isos since $\tilde{K}(D_+^{2n}) = 0$

excision

$\{e\} \xrightarrow{\approx} D^{2n}$

$f| : D^{2n} \times \{e\} \xrightarrow{\approx} D_-^{2n}$ a homeomorphism from which we deduce that $\beta \in \tilde{K}(C_f)$ corresponds to a generator of $\tilde{K}(S^{2n})$ under the maps in the left-hand column, and similarly with +/- exchanged.

Now the commutative diagram

$$
\begin{tikzcd}
\beta\otimes\beta \arrow[rr, mapsto] & & \beta^2 & \\
\tilde{K}(C_f)\otimes\tilde{K}(C_f) \arrow[rr, "\text{prod.}"] & & \tilde{K}(C_f) & \alpha \\
K(C_f, D^{2n}_+)\otimes K(C_f, D^{2n}_-) \arrow[u, "\approx"] \arrow[d, "\Phi^*\otimes\Phi^*"] \arrow[rr, "\text{prod.}"] & & K(C_f, S^{2n}) \arrow[u] & \text{gen.} \arrow[u, mapsto] \\
K((D^{2n},\partial D^{2n})\times D^{2n})\otimes K(D^{2n}\times(D^{2n},\partial D^{2n})) \arrow[rr, "\text{prod.}"] \arrow[d, "\approx"', "(\text{incl})^*\otimes(\text{incl})^*"] & & K(D^{2n}\times D^{2n}, \partial(D^{2n}\times D^{2n})) \arrow[u, "\approx"', "\Phi^*"] & \\
K((D^{2n},\partial D^{2n})\times\{e\})\otimes K(\{e\}\times(D^{2n},\partial D^{2n})) \arrow[urr, "*"] & & \tilde{K}(S^{4n}) \arrow[u, "\approx"', "(\text{quot})^*"] & \\
\tilde{K}(S^{2n})\otimes\tilde{K}(S^{2n}) \arrow[u, "\approx"', "(\text{quot})^*\otimes(\text{quot})^*"] \arrow[urr, "*"', "\approx"] & & & \\
\text{gen}\otimes\text{gen} \arrow[uuuuuu, mapsto, bend left] & & &
\end{tikzcd}
$$

Prop $\boxed{XXI}$. 8

implies that $\beta^2 = \pm\alpha$. □

In view of Lemma 3, Prop. 4 and Lemma 7, Thm 1 follows from

<u>Thm 8</u> (Adams): If $f: S^{4n-1} \longrightarrow S^{2n}$ has Hopf invariant ± 1, then $n = 1, 2$ or 4.

This is what we will prove.