

Division algebras and parallelizability of spheres (cont.)

Recall that we want to show

Thm 1:

(i) If \mathbb{R}^n admits the structure of a division algebra, then $n=1, 2, 4$ or 8 .

(ii) If S^n is parallelizable, then $n=0, 1, 3$ or 7 .

H-spaces and the Hopf invariant

Def 2: An H-space is a space X equipped with a continuous multiplication $\mu: X \times X \rightarrow X$ which has a two-sided identity element $e \in X$.

Note: There is some variation in the definition of an H-space in the literature: often, it is only assumed that $\mu(e, -)$ and $\mu(-, e)$ are homotopic to id_X instead of being equal to it. Under mild conditions on X and e , the two definitions are equivalent (in the sense that a μ satisfying the laxer definition can be homotoped to one satisfying the stricter one).

Lemma 3: If \mathbb{R}^n is a division algebra on S^{n-1} is parallelizable, then S^{n-1} is an H-space.

Pf: Suppose \mathbb{R}^n is a division algebra. By Lemma XXII.8, we may assume that it is unital.

Now

$$\begin{array}{ccc} S^{n-1} \times S^{n-1} & \longrightarrow & S^{n-1} \\ (x, y) & \longmapsto & x \cdot y / |x \cdot y| \end{array}$$

is an H-space structure on S^{n-1} .

So, now S^{n-1} is parallelizable. Then \exists linearly independent sections

$$s_1, \dots, s_{n-1} : S^{n-1} \longrightarrow TS^{n-1}$$

The vectors $s_1(e_1), \dots, s_{n-1}(e_1)$ form a basis of $T_{e_1} S^{n-1} = \text{span}(e_2, \dots, e_n) \subset \mathbb{R}^n$. Replacing s_1, \dots, s_{n-1} by suitable linear combinations, we may assume that

$$s_1(e_1) = e_2, \dots, s_{n-1}(e_1) = e_n.$$

Using the Gram-Schmidt process, we may further assume that the vectors

$$x, s_1(x), \dots, s_{n-1}(x)$$

are orthonormal for all $x \in S^{n-1}$; notice that the Gram-Schmidt process does not change $s_1(e_1), \dots, s_{n-1}(e_1)$, since the vectors $e_1, s_1(e_1), \dots, s_{n-1}(e_1)$ are already orthonormal. For $x \in S^{n-1}$, let $\alpha_x \in O(n)$ be the element mapping the standard basis to $x, s_1(x), \dots, s_{n-1}(x)$. Now $(x, y) \mapsto \alpha_x(y)$ is an H-space structure on S^{n-1} with identity element e_1 : we have $\alpha_{e_1} = \text{id}$ and $\alpha_x(e_1) = x$ for all $x \in S^{n-1}$. \square

By Lemma 3, to prove Thm 1, it is enough to show that if S^{n-1} is an H-space, then $n=1, 2, 4$ or 8 . The first step is to show that even-dimensional spheres are not H-spaces (except for S^0):

Prop 4: S^{2n} is not an H-space for any $n > 0$.

Pf: Sp. $\mu: S^{2n} \times S^{2n} \rightarrow S^{2n}$ is an H-space multiplication with identity element $e \in S^{2n}$. Let

$$i_1: S^{2n} \rightarrow S^{2n} \times S^{2n}, \quad i_1(x) = (x, e)$$

$$i_2: S^{2n} \rightarrow S^{2n} \times S^{2n}, \quad i_2(x) = (e, x)$$

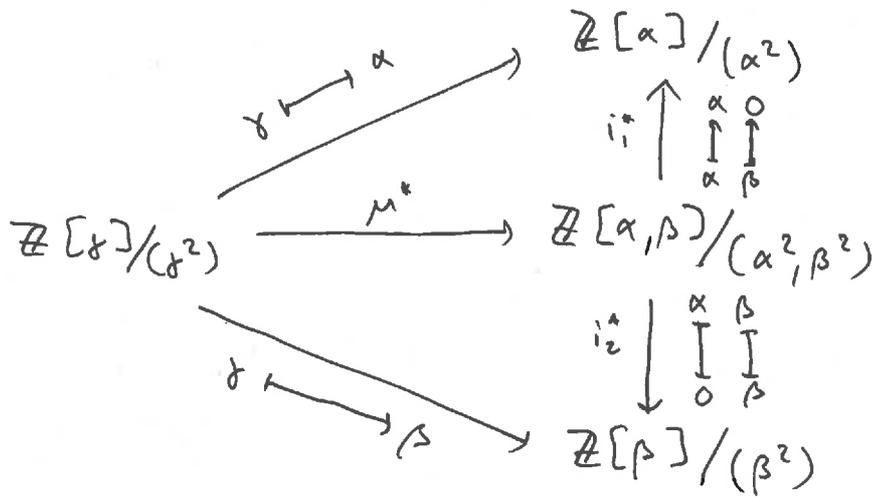
$$i: pt \rightarrow S^{2n}, \quad pt \mapsto e$$

Then the following diagram commutes:

$$\begin{array}{ccccc}
 & & K(S^{2n}) & \xleftarrow[\cong]{*} & K(S^{2n}) \otimes K(pt) \\
 & \nearrow id & \uparrow i_1^* & & \uparrow 1 \otimes i^* \\
 K(S^{2n}) & \xrightarrow{\mu^*} & K(S^{2n} \times S^{2n}) & \xleftarrow[\cong]{*} & K(S^{2n}) \otimes K(S^{2n}) \\
 & \searrow id & \downarrow i_2^* & & \downarrow i^* \otimes 1 \\
 & & K(S^{2n}) & \xleftarrow[\cong]{*} & K(pt) \otimes K(\cancel{S^{2n}})
 \end{array}$$

The horizontal maps on the right are isomorphisms by Prop. XXI.9.

In view of the computation $K(S^{2n}) = \mathbb{Z}[x]/(x^2)$ (Cor. XXI.3) and the observation that $i^*(x) = 0$ (since $x \in \tilde{K}(S^{2n})$), the part on the left is



It follows that $\mu^*(\delta) = \alpha + \beta + m\alpha\beta$ for some $m \in \mathbb{Z}$. Therefore

$$\mu^*(\delta^2) = \mu^*(\delta)^2 = (\alpha + \beta + m\alpha\beta)^2 = 2\alpha\beta \neq 0,$$

a contradiction since $\delta^2 = 0$. \square

For a map $f: S^{4n-1} \rightarrow S^{2n}$, write C_f for the space $S^{2n} \cup_f D^{4n}$. Since $\tilde{K}^1(S^{2n}) = \tilde{K}^1(S^{4n}) = 0$, the long exact sequence for

$$S^{2n} \xrightarrow{i} C_f \xrightarrow{q} S^{4n}$$

gives a short exact sequence

$$0 \rightarrow \tilde{K}(S^{4n}) \xrightarrow{q^*} \tilde{K}(C_f) \xrightarrow{i^*} \tilde{K}(S^{2n}) \rightarrow 0$$

$\cong \mathbb{Z} \qquad \qquad \qquad \cong \mathbb{Z}$

Let $\alpha = q^*((H-1)^{*2n}) \in \tilde{K}(C_f)$

and let $\beta \in \tilde{K}(C_f)$ be s.t. $i^*(\beta) = (H-1)^{*n} \in \tilde{K}(S^{2n})$.

Since $i^*(\beta^2) = (i^*\beta)^2 = 0 \in \tilde{K}(S^{2n})$, we have

$$\beta^2 = h\alpha$$

for some $h \in \mathbb{Z}$.

Def 5: The integer h is the Hopf invariant of the map $f: S^{4n-1} \rightarrow S^{2n}$.

Lemma 6: The integer h is independent of the choice of β .

Pf: Any other choice of β is of the form $\beta + m\alpha$ for some $m \in \mathbb{Z}$. We have

$$(\beta + m\alpha)^2 = \beta^2 + 2m\alpha\beta + m^2\alpha^2 = \beta^2 + 2m\alpha\beta,$$

so it is enough to show that $\alpha\beta = 0$. Since $i^*(\alpha\beta) = i^*(\alpha)i^*(\beta) = 0$, we have $\alpha\beta = k\alpha$ for some $k \in \mathbb{Z}$. Now

$$k^2\alpha = k\alpha\beta = \alpha\beta^2 = h\alpha^2 = 0.$$

Since α generates an infinite cyclic group, it follows that $k^2 = 0$, so $k = 0$ and $\alpha\beta = k\alpha = 0$. \square

Lemma 7: If S^{2n-1} admits the structure of an H-space, then \exists map $f: S^{4n-1} \rightarrow S^{2n}$ of Hopf invariant ± 1 .

Pf: Let $\mu: S^{2n-1} \times S^{2n-1} \rightarrow S^{2n-1}$ be an H-space multiplication with neutral element $e \in S^{2n-1}$.

View S^{2n} as a union of two disks along the equator,

$$S^{2n} = D_+^{2n} \cup D_-^{2n}$$

and S^{4n-1} as

$$S^{4n-1} = \partial(D_+^{2n} \times D_-^{2n}) = \partial D_+^{2n} \times D_-^{2n} \cup D_+^{2n} \times \partial D_-^{2n}$$

The maps

$$\begin{aligned} \partial D_+^{2n} \times D_-^{2n} &\longrightarrow D_+^{2n} \\ (x, y) &\longmapsto |y| \mu(x, y/|y|) \end{aligned}$$

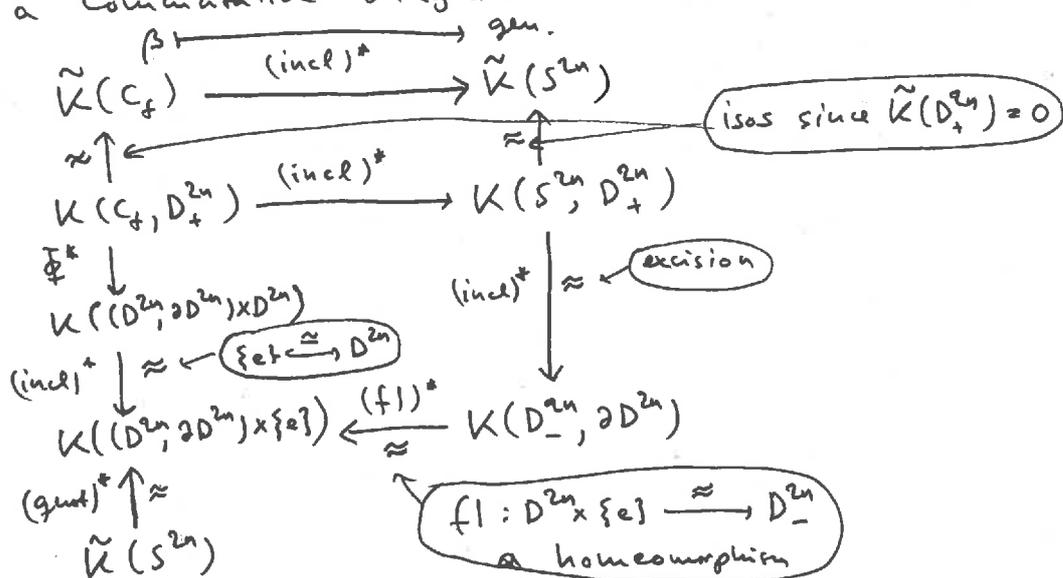
$$\begin{aligned} D_+^{2n} \times \partial D_-^{2n} &\longrightarrow D_-^{2n} \\ (x, y) &\longmapsto |x| \mu(x/|x|, y) \end{aligned}$$

make sense and are continuous even for $x=0$ and $y=0$, and they fit together to define a map

$$f: S^{4n-1} \longrightarrow S^{2n}$$

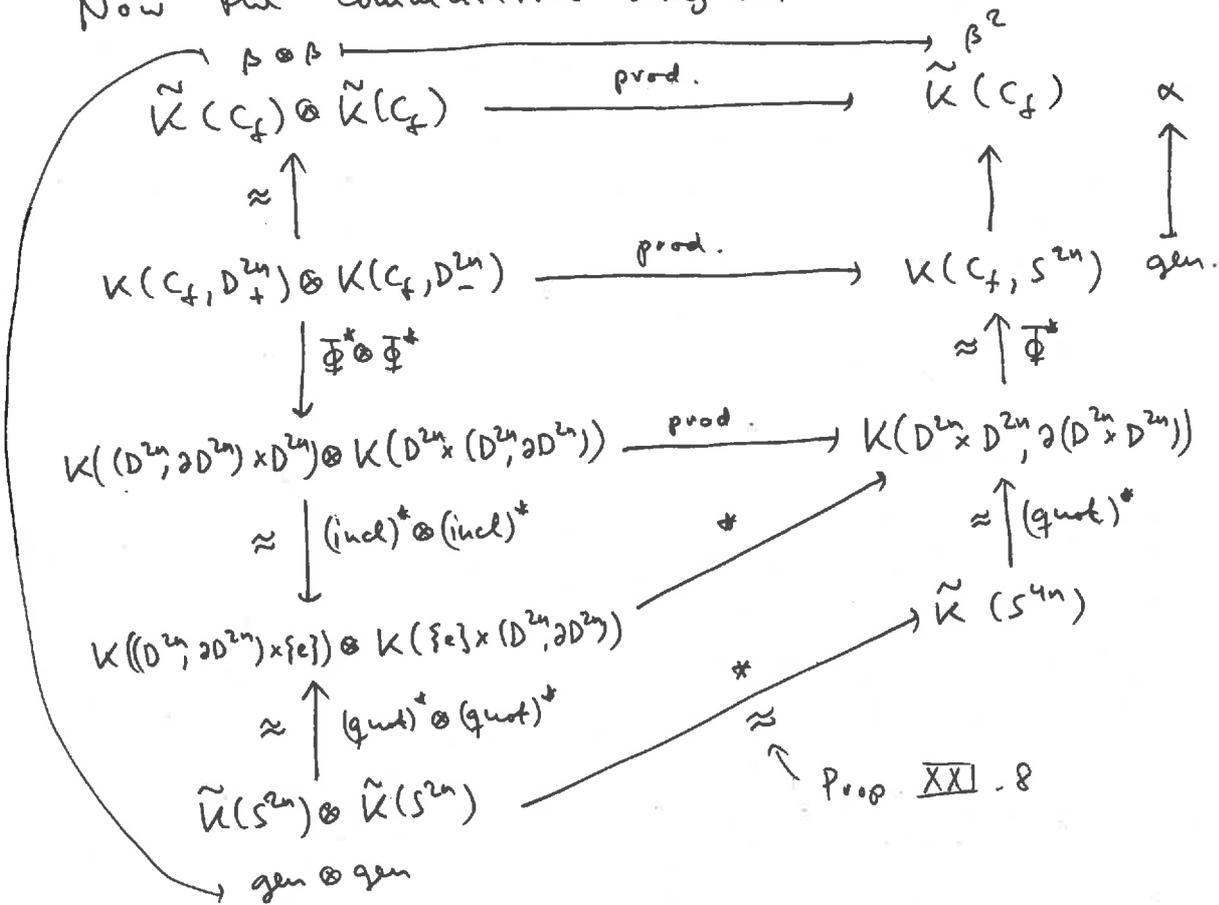
Let $\Phi: (D_+^{4n}, S^{4n-1}) = (D_+^{2n} \times D_-^{2n}, \partial(D_+^{2n} \times D_-^{2n})) \longrightarrow (C_f, S^{2n})$

be the evident map $D_+^{4n} \longrightarrow C_f = S^{2n} \cup_f D_+^{4n}$. We have a commutative diagram



from which we deduce that $\beta \in \tilde{K}(C_f)$ corresponds to a generator of $\tilde{K}(S^{2n})$ under the maps in the left-hand column, and similarly with $+/-$ exchanged.

Now the commutative diagram



implies that $\beta^2 = \pm \alpha$. \square

In view of Lemma 3, Prop. 4 and Lemma 7, Thm 1 follows from

Thm 8 (Adams): If $f: S^{4n-1} \rightarrow S^{2n}$ has Hopf invariant ± 1 , then $n = 1, 2$ or 4 .

This is what we will prove.