Adams operations

Recall : We are trying to that if $\mathbb{R}^n$ is a division algebra, then $n=1,2,4$ or 8, and that if S^n is parallelizable, then $n=0,1,3$ or 7. Last time, we reduced this task to proving

Thm 1 (Adams): If $S^{4n-1} \longrightarrow S^{2n}$ has Hopf invariant ± 1, then $n=1,2$ or 4.

The theorem was first proven by Adams using secondary cohomology operations (~ 1960). The much simpler K-theory proof we are going to discuss was discovered by Atiyah (~ 1966). The proof is based on Adams operations

$$\psi^k : K(X) \longrightarrow K(X).$$

Thm 2 (Adams operations): For X a compact Hausdorff space, there exist ring homomorphisms

$$\psi^k : K(X) \longrightarrow K(X), \quad k \in \mathbb{Z}_{\geq 0},$$

satisfying

(1) $\psi^k f^* = f^* \psi^k$ for all $f : X \to Y$

(2) $\psi^k(L) = L^k$ if L is a line bundle

(3) $\psi^k \circ \psi^l = \psi^{kl}$

(4) $\psi^p(\alpha) \equiv \alpha^p$ mod p if p is a prime (i.e. $\psi^p(\alpha) - \alpha^p = p\beta$ for some $\beta \in K(X)$).

<u>Idea of the construction</u>: For $\xi = L_1 \oplus \cdots \oplus L_n$ a sum of line bundles, we must have

$$\psi^k(\xi) = L_1^k + \cdots + L_n^k .$$

It turns out that the RHS can be expressed as a polynomial in the exterior powers of ξ. This expression makes sense even when ξ is not a sum of line bundles, so we can use it to define $\psi^k(\xi)$ in general.

<u>Interlude: symmetric polynomials</u>

<u>Def 3</u>: A polynomial $p(x_1, \dots, x_n)$ in variables $x_1, \dots, x_n$ is <u>symmetric</u> if

$$p(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = p(x_1, \dots, x_n)$$

for all permutations $\sigma \in \Sigma_n$.

<u>Eg</u>: The polynomial $p_k = p_k^{(n)} = x_1^k + \cdots + x_n^k$ is symmetric.

<u>Eg</u>: Since $\prod_{i=1}^{n}(1+x_i)$ is symmetric, so is its degree i homogeneous part $\sigma_i = \sigma_i^{(n)}$, the i-th <u>elementary symmetric polynomial</u> in $x_1, \dots, x_n$. Explicitly,

$$\sigma_0^{(n)} = 1, \quad \sigma_1^{(n)} = x_1 + \cdots + x_n, \quad \sigma_2^{(n)} = \sum_{1 \le i < j \le n} x_i x_j, \dots, \quad \sigma_n^{(n)} = x_1 x_2 \cdots x_n .$$

Notice that $\sigma_i^{(n)} = 0$ for $i > n$ and that

$$\sigma_i^{(n)}(x_1, \dots, x_n) = \sigma_i^{(n+1)}(x_1, \dots, x_n, 0).$$

It is a fundamental result concerning symmetric polynomials that every symmetric polynomial can be expressed uniquely as a polynomial in the elementary symmetric polynomials. For us, however, the following result is sufficient:

Prop 4 (Newton's identity): For all $k, n \geq 1$,

$$p_k^{(n)} = (-1)^{k-1} k \sigma_k^{(n)} + \sum_{i=1}^{k-1} (-1)^{k-1-i} \sigma_{k-i}^{(n)} p_i^{(n)}.$$

Pf: Substituting $t = -x_j$ to the identity

$$\prod_{i=1}^{k} (t + x_i) = \sum_{i=0}^{k} \sigma_{k-i}^{(k)} t^i$$

gives

$$0 = \sum_{i=0}^{k} (-1)^i \sigma_{k-i}^{(k)} x_j^i .$$

Summing over j gives the claim for $n = k$. For $n < k$, the claim follows by substituting $x_{n+1} = \cdots = x_k = 0$. For $n > k$, notice that each side is a sum of monomials involving at most k of $x_1, \ldots, x_n$. Substituting 0 for $n-k$ of the other variables, it follows from the case $n = k$ that the coefficients on both sides for such monomials agree. The claim follows. □

By recursively applying Newton's identity, we see that there is a polynomial s_k with integer coefficients, the k-th Newton polynomial, s.t.

$$s_k(\sigma_1^{(n)}, \ldots, \sigma_k^{(n)}) = p_k^{(n)} = x_1^k + \cdots + x_n^k$$

for all $n \geq 1$.

Interlude: exterior powers

Let $\mathbb{F}$ be a field (such as $\mathbb{R}$ or $\mathbb{C}$)

Def 5: The k-th exterior power of an $\mathbb{F}$-vector space V is the quotient

$$\Lambda^k V = T^k V / I_k$$

where $T^k V = V^{\otimes k}$ $(T^0 V = \mathbb{F})$ and $I_k \subset V^{\otimes k}$ is the subspace generated by tensors $v_1 \otimes \cdots \otimes v_k$ s.t. $v_i = v_{i+1}$ for some $1 \leq i < k$.

Idea: For an $\mathbb{F}$-vector space X,

$$\{\text{linear maps } T^k V \to X\} \leftrightarrow \{k\text{-linear maps } V^k \to X\}$$

$$\{\text{linear maps } \Lambda^k V \to X\} \leftrightarrow \{\text{alternating } k\text{-linear maps } V^k \to X\}.$$

Recall that a k-linear map $f: V^k \to X$ is alternating if $f(v_1, \ldots, v_k) = 0$ whenever $v_i = v_{i+1}$ for some $1 \leq i < k$.

Properties: there are natural isomorphisms

(1) $\Lambda^0(V) \approx \mathbb{F}$

(2) $\Lambda^1(V) \approx V$

(3) $\Lambda^k(V) = 0$ for $k > \dim(V)$

(4) $\Lambda^k(V \oplus W) \approx \bigoplus_{i+j=k} \Lambda^i(V) \otimes \Lambda^j(W)$

for V, W finite-dimensional $\mathbb{F}$-vector spaces.

For more details, see the separate handout.

Adams operations, continued

As usual, the construction of exterior powers and the aforementioned isomorphisms generalize to vector bundles. For a complex v.b. $\xi \to X$, write

$$\lambda^k(\xi) = \Lambda^k(\xi)$$
$$\lambda_t(\xi) = \sum_{k\geq 0} \lambda^k(\xi) t^k \in K(X)[t].$$

By (3), the sum in the definition of $\lambda_t(\xi)$ is finite. By (4), we have

$$\lambda_t(\xi \oplus \zeta) = \lambda_t(\xi)\lambda_t(\zeta),$$

and by (1), (2) and (3) we have

$$\lambda_t(L) = 1 + Lt$$

when $L \to X$ is a line bundle. Thus for $\xi = L_1 \oplus \cdots \oplus L_n$ a sum of line bundles, we have

$$\lambda_t(\xi) = \prod_{i=1}^n (1 + L_i t).$$

Therefore in this case

$$\lambda^k(\xi) = \sigma_k^{(n)}(L_1, \ldots, L_n)$$

and hence

$$s_k(\lambda^1(\xi), \ldots, \lambda^k(\xi)) = L_1^k + \cdots + L_n^k,$$

which is what $\psi^k(\xi)$ should be.

Def 6: For a complex v.b. $\xi \to X$, we define

$$\psi^k(\xi) := s_k(\lambda^1(\xi), \dots, \lambda^k(\xi)).$$

To proceed, we need the following key result whose proof we will postpone until later.

Thm 7 (Splitting principle): Sp. X is a compact Hausdorff space and let $\xi \to X$ be a complex v.b. Then $\exists$ compact Hausdorff space Y and a map $p: Y \to X$ s.t.

$$p^*: K(X) \longrightarrow K(Y)$$

is injective and $p^*(\xi) \to Y$ splits as a sum of line bundles.

Pf of Thm 2: So far, we have defined a map

$$\mathrm{Vect}_{\mathbb{C}}(X) \xrightarrow{\psi^k} K(X).$$

Since $f^*\lambda^k(\xi) \approx \lambda^k(f^*\xi)$, we have

$$(*) \qquad f^*\psi^k(\xi) = \psi^k(f^*\xi)$$

for f a map to the base of ξ. We claim that

$$\psi^k(\xi \oplus \zeta) = \psi^k(\xi) + \psi^k(\zeta).$$

By $(*)$ and a repeated application of the splitting principle, it suffices to prove this when ξ, ς split as sums

$$(**) \qquad \begin{aligned} \xi &= L_1 \oplus \cdots \oplus L_n \\ \varsigma &= L_1' \oplus \cdots \oplus L_{n'}' \end{aligned}$$

of line bundles. Now

$$\begin{aligned} \psi^k(\xi \oplus \varsigma) &= \psi^k(L_1 \oplus \cdots \oplus L_n \oplus L_1' \oplus \cdots \oplus L_{n'}') \\ &= L_1^k + \cdots + L_n^k + (L_1')^k + \cdots + (L_{n'}')^k \\ &= \psi^k(\xi) + \psi^k(\varsigma), \end{aligned}$$

as claimed. It follows that $\psi^k : \mathrm{Vect}_{\mathbb{C}}(X) \to K(X)$ induces a homomorphism

$$\psi^k : K(X) \longrightarrow K(X)$$

of abelian groups. Properties (1) and (2) in Thm 2 are clear. To see that ψ^k preserves products, it is enough to verify that

$$\psi^k(\xi \otimes \varsigma) = \psi^k(\xi)\, \psi^k(\varsigma)$$

for v.b.'s $\xi, \varsigma \to X$. By the splitting principle, we may again assume that ξ, ς split as sums of line bundles as in $(**)$, in which case

$$\begin{aligned} \psi^k(\xi \otimes \varsigma) &= \psi^k(\oplus_{i,j}(L_i \otimes L_j')) \\ &= \Sigma_{i,j}(L_i \otimes L_j')^k \\ &= \Sigma_{i,j} L_i^k (L_j')^k \end{aligned}$$

$$= \left(\Sigma_i L_i^k\right)\left(\Sigma_j (L_j')^k\right)$$
$$= \psi^k(\xi)\,\psi^k(\zeta),$$

as desired.

Property (3) follows from the splitting principle, additivity, and the computation

$$\psi^k\psi^\ell(L) = \psi^k(L^\ell) = (L^\ell)^k = L^{k\ell} = \psi^{k\ell}(L)$$

for L a line bundle.

(4) follows from the splitting principle and the computation that for $\xi = \oplus_{i=1}^n L_i$,

$$\begin{aligned}\psi^p(\xi) &= L_1^p + \cdots + L_n^p \\ &\equiv (L_1 + \cdots + L_n)^p \\ &= \xi^p \qquad (\mathrm{mod}\ p). \qquad \square\end{aligned}$$