Adams operations (cont.)

Recall: we want to prove

Thm 1 (Adams): If $S^{4n-1} \longrightarrow S^{2n}$ has Hopf invariant ± 1, then $n = 1, 2$ or 4.

Last time, we constructed Adams operations (modulo the proof of the splitting principle):

Thm 2 (Adams operations): For X a compact Hausdorff space, there are ring homomorphisms

$$\psi^k : K(X) \longrightarrow K(X), \quad k \geq 1,$$

satisfying

(1) $\psi^k f^* = f^* \psi^k$ for all $f: X \to Y$

(2) $\psi^k(L) = L^k$ if L is a line bundle

(3) $\psi^k \circ \psi^\ell = \psi^{k\ell}$

(4) $\psi^p(\alpha) \equiv \alpha^p \mod p$ if p is a prime. $\square$

By naturality, we also get operations in reduced K-theory:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{K}(X) \arrow[r] \arrow[d, dashed, "\psi^k"'] & K(X) \arrow[r] \arrow[d, "\psi^k"] & K(pt) \arrow[r] \arrow[d, "\psi^k"] & 0 \\
0 \arrow[r] & \tilde{K}(X) \arrow[r] & K(X) \arrow[r] & K(pt) \arrow[r] & 0
\end{tikzcd}$$

Unrolling the def'n of the reduced external product, we see that

$$\begin{tikzcd}
\tilde{K}(X) \otimes \tilde{K}(Y) \arrow[r, "*"] \arrow[d, "\psi^k \otimes \psi^k"'] & \tilde{K}(X \wedge Y) \arrow[d, "\psi^k"] \\
\tilde{K}(X) \otimes \tilde{K}(Y) \arrow[r, "*"] & \tilde{K}(X \wedge Y)
\end{tikzcd}$$

commutes, i.e.

$$\psi^k(x * y) = \psi^k(x) * \psi^k(y).$$

<u>Prop 3</u>: $\psi^k : \tilde{K}(S^{2n}) \longrightarrow \tilde{K}(S^{2n})$ is multiplication by k^n.

<u>Pf</u>: We have

$$\begin{aligned}
\psi^k(H-1) &= H^k - 1 \\
&= (1 + (H-1))^k - 1 \\
&= 1 + k(H-1) - 1 \\
(H-1)^2 = 0 \quad &= k\,(H-1)
\end{aligned}$$

So

$$\begin{aligned}
\psi^k(\underbrace{(H-1)^{*n}}_{\text{generator of } \tilde{K}(S^{2n})}) &= (\psi^k(H-1))^{*n} \\
&= (k(H-1))^{*n} \\
&= k^n (H-1)^{*n}.
\end{aligned}$$

$\square$

<u>Proof of Thm 1</u>: Sp. $f : S^{4n-1} \to S^{2n}$ has Hopf invariant ± 1. Let $\alpha, \beta \in \tilde{K}(C_f)$ be as in the definition of the Hopf invariant:

$$0 \to \tilde{K}(S^{4n}) \longrightarrow \tilde{K}(C_f) \longrightarrow \tilde{K}(S^{2n}) \longrightarrow 0$$
$$(H-1)^{*2n} \longmapsto \alpha \qquad \beta \longmapsto (H-1)^{*n}$$

By Prop 3, we have

$$\psi^k(\alpha) = k^{2n}\alpha$$
$$\psi^k(\beta) = k^n\beta + \mu_k\alpha \qquad \text{for some } \mu_k \in \mathbb{Z}.$$

In particular,

$$\psi^2(\beta) = 2^n\beta + \mu_2\alpha$$

Since $\psi^2(\beta) \equiv \beta^2 \pmod 2$ by property (4) and $\beta^2 = \pm\alpha$ by the assumption that f has Hopf invariant ± 1, μ_2 must be odd. We have

$$\psi^6(\beta) = \psi^3\psi^2(\beta) = 6^n\beta + (2^n\mu_3 + 3^{2n}\mu_2)\alpha$$
$$\psi^6(\beta) = \psi^2\psi^3(\beta) = 6^n\beta + (2^{2n}\mu_3 + 3^n\mu_2)\alpha,$$

so $\quad 2^n\mu_3 + 3^{2n}\mu_2 = 2^{2n}\mu_3 + 3^n\mu_3$, or

$$3^n(3^n - 1)\mu_2 = 2^n(2^n-1)\mu_3.$$

Since μ_2 is odd, it follows that $2^n \mid 3^n - 1$.

The proof is now completed by the following lemma. □

Lemma 4: If $2^n \mid 3^n - 1$, then $n = 1, 2$ or 4.

Pf: Sp. $2^n \mid 3^n - 1$. If n is odd, then

$3^n - 1 \equiv (-1)^n - 1 \equiv 2 \pmod 4$, so $n = 1$.

Sp. $n = 2k$ for some $k \geq 1$. Then

$$3^n - 1 = (3^k - 1)(3^k + 1).$$

One of the numbers $3^k - 1$, $3^k + 1$ is congruent to $2 \pmod 4$, so we have

$$2^{2k-1} \mid 3^k - 1 \quad \text{or} \quad 2^{2k-1} \mid 3^k + 1.$$

If $k \geq 3$, then $2^{2k-1} > 3^k \pm 1$, so $k = 1$ or $k = 2$, i.e. $n = 2$ or $n = 4$. □

This concludes the proof of Adams's theorem — modulo the proof of the splitting principle.

The splitting principle

Def 5: A fibre bundle with base space B, total space E and fibre F is a continuous map $p : E \to B$ s.t. for every $b \in B$, $\exists$ neighbourhood U of b and a homeomorphism (local trivialization) $h : p^{-1}U \longrightarrow U \times F$ making the triangle

$$\begin{tikzcd} p^{-1}U \arrow[rr, "h", "\approx"'] \arrow[dr, "p|"'] & & U \times F \arrow[dl, "pr"] \\ & U & \end{tikzcd}$$

commute.

<u>Eg</u>: An $\mathbb{F}$-v.b. $\xi \to X$ of constant dimension n is a fibre bundle with base space X, total space ξ, and fibre $\mathbb{F}^n$.

<u>Eg</u>: For $\xi \to X$ as above, write $P(\xi)$ for the quotient

$$P(\xi) = \{v \in \xi \mid v \neq 0\}/\sim$$

where $\sim$ identifies v with λv for all $\lambda \in \mathbb{F}^\times$.
Then $\exists$ evident map

$$p : P(\xi) \longrightarrow X$$

and p is a fibre bundle with fibre $\mathbb{F}P^{n-1}$.
Local trivializations

$$\xi|U \xrightarrow{\approx} U \times \mathbb{F}^n$$

induce local trivializations

$$P(\xi)|U \xrightarrow{\approx} U \times \mathbb{F}P^{n-1}.$$

Intuition: $P(\xi)$ is obtained from ξ by replacing each fibre ξ_b by the projective space $P(\xi_b)$ of lines in ξ_b.

The splitting principle will follow from

<u>Thm 6</u> (Leray - Hirsch): Let $p: E \to B$ be a fibre bundle with E, B compact Hausdorff and fibre F a finite cell complex with only even-dimensional cells. Sp. there exist classes $c_1, \dots, c_k \in K(E)$ which restrict to a basis for $K(E_b)$ for each fibre $E_b = p^{-1}(b)$. Then $K^*(E)$ is a free $K^*(B)$-module with basis $\{c_1, \dots, c_k\}$.

The $K^*(B)$-module structure on $K^*(E)$ is

$$\beta \cdot \gamma = p^*(\beta)\gamma \quad \text{for } \beta \in K^*(B),\ \gamma \in K^*(E).$$

We will need

<u>Lemma 7</u> (Five lemma): Sp.

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

is a commutative diagram of abelian groups with exact rows s.t. α is epi, β and δ are isos, and ε is a mono. Then γ is an iso.

<u>Pf</u>: Exercise. $\square$

Lemma 8: Let Y be a finite cell complex with only even-dimensional cells, and choose one of the 0-cells as a basepoint. Then for any pointed compact Hausdorff space X, the external product

$$\tilde{K}^*(X) \otimes \tilde{K}^*(Y) \xrightarrow{*} \tilde{K}^*(X \wedge Y)$$

is an isomorphism.

Pf: We proceed by induction on the number of cells in Y. For $Y = pt$, both the source and the target are 0, so the claim holds. So now Y is obtained from a subcomplex B containing the basepoint by attaching a single $2n$-cell. The sequence

$$B \xhookrightarrow{i} Y \xrightarrow{q} Y/B$$

gives a split exact sequence

$$0 \to \tilde{K}^*(Y/B) \xrightarrow{q^*} \tilde{K}^*(Y) \xrightarrow{i^*} \tilde{K}^*(B) \to 0;$$

indeed, by Prop XXII.1 the groups are free and concentrated in degree 0. Tensoring with $\tilde{K}^*(X)$ gives the exact sequence on top of the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{K}^*(X) \otimes \tilde{K}^*(Y/B) \arrow[r, "1 \otimes q^*"] \arrow[d, "*", "\text{Bott periodicity } \approx"'] & \tilde{K}^*(X) \otimes \tilde{K}^*(Y) \arrow[r, "1 \otimes i^*"] \arrow[d, "*"] & \tilde{K}^*(X) \otimes \tilde{K}^*(B) \arrow[r] \arrow[d, "*"', "\approx \leftarrow \text{induction}"] & 0 \\
& \tilde{K}^*(X \wedge Y/B) \arrow[r, "(1 \wedge q)^*"] & \tilde{K}^*(X \wedge Y) \arrow[r, "(1 \wedge i)^*"] & \tilde{K}^*(X \wedge B) &
\end{tikzcd}$$

By commutativity of the diagram, the map $(1 \wedge i)^*$ is an epi. From the exact sequence for

$$\mathbb{X} \wedge B \hookrightarrow \mathbb{X} \wedge Y \longrightarrow \mathbb{X} \wedge Y / \mathbb{X} \wedge B$$

$$\mathbb{X} \wedge Y / \mathbb{X} \wedge B = \mathbb{X} \wedge Y/B$$

we deduce that the bottom row is exact, and that $(1 \wedge q)^*$ is a mono. The claim now follows from the five lemma (with $A, A', E, E' = 0$). $\square$