The splitting principle (cont.)

As a step towards proving the splitting principle, we are trying to prove

Thm 1 (Leray-Hirsch): Let $p: E \to B$ be a fibre bundle with E, B compact Hausdorff and fibre F a finite cell complex with only even-dimensional cells. Suppose there exist classes $c_1, \ldots, c_k \in K(E)$ which restrict to a basis for $K(E_b)$ for each fibre $E_b = p^{-1}(b)$ of $p: E \to B$. Then $K^*(E)$ is a free $K^*(B)$-module with basis $\{c_1, \ldots, c_k\}$.

In other words, the map

$$\begin{aligned} \Phi : K^*(B) \otimes K^*(F) &\longrightarrow K^*(E) \\ \textstyle\sum_{i=1}^k b_i \otimes i^*(c_i) &\longmapsto \textstyle\sum_{i=1}^k p^*(b_i) c_i \end{aligned}$$

is an isomorphism. (Here $i: F \hookrightarrow E$ is the inclusion of some fibre.)

We will first prove the theorem in the case where E is trivial, i.e. $E = B \times F$, $p = pr: B \times F \to B$.

Def 2: For X a compact Hausdorff space, write $H^0(X)$ for the ring of locally constant functions $X \to \mathbb{Z}$ (under pointwise operations). The map

$$\begin{aligned} \mathrm{Vect}_{\mathbb{C}}(X) &\longrightarrow H^0(X) \\ \xi &\longmapsto (x \mapsto \dim(\xi_x)) \end{aligned}$$

induces a ring epimorphism

$$K(X) \longrightarrow H^0(X)$$

Write $\hat{K}(X)$ for the kernel of this homomorphism.
(The notation is not standard.)

Eg: If X is connected, then $\hat{K}(X) = \tilde{K}(X)$.

Recall: An element x of a ring R is nilpotent if $x^n = 0$ for some $n > 0$.

Lemma 3: For any compact Hausdorff space X, the elements of $\hat{K}(X)$ are nilpotent.

Pf: For a closed subspace $A \subset X$, write $K_A(X)$ for the kernel of the map

$$K(X) \xrightarrow{(\mathrm{incl})^*} K(A)$$

or, what is the same, for the image of the map

$$K(X, A) \xrightarrow{(\mathrm{incl})^*} K(X).$$

Then $K_{A_1}(X) K_{A_2}(X) \subset K_{A_1 \cup A_2}(X)$ by the commutativity of

$$\begin{array}{ccc}
K(X, A_1) \otimes K(X, A_2) & \longrightarrow & K(X, A_1 \cup A_2) \\
\Big\downarrow {\scriptstyle (\mathrm{incl})^* \otimes (\mathrm{incl})^*} & & \Big\downarrow {\scriptstyle (\mathrm{incl})^*} \\
K(X) \otimes K(X) & \longrightarrow & K(X)
\end{array}$$

So now $\alpha \in \hat{K}(X)$. By Rk XII.1, we have
$$\alpha = \xi - \varepsilon^k$$
for some v.b. $\xi \to X$. Since X is compact, we can find a finite number of closed subsets
$$A_1, \dots, A_n \subset X$$
s.t. $\xi | A_i$ is trivial for all $i = 1, \dots, n$. Since $\alpha \in \hat{K}(X)$, the class α restricts to 0 in $K(A_i)$ for all i, so $\alpha \in K_{A_i}(X)$ for $i = 1, \dots, n$. Thus
$$\alpha^n \in K_{A_1 \cup \dots \cup A_n}(X) = K_X(X) = 0,$$
so α is nilpotent. $\square$

For $\varphi : R \to S$ a ring homomorphism between commutative rings and A a matrix with coefficients in R, write φA for the matrix with coefficients in S obtained by applying φ to A entrywise. If A is a square matrix, notice that

$$(*) \qquad \det(\varphi A) = \varphi \det(A).$$

<u>Lemma 4</u>: Let $\pi : R \to S$ be an epimorphism of commutative rings s.t. $\ker(\pi)$ consists of nilpotent elements. Then a square matrix A with coefficients in R is invertible iff πA is.

<u>Pf</u>: "$\Rightarrow$" is immediate from $(*)$.
"$\Leftarrow$": Let $u = \det(A)$. Then $\pi(u) = \det(\pi A) \in S^\times$, so we can find $u' \in R$ s.t. $\pi(u)\pi(u') = 1$ in S.

Then $uu'-1 \in \mathrm{Ker}(\pi)$, so $(uu'-1)^n = 0$ for some $n > 0$. Now

$$u \left(\Sigma_{i=1}^{n} (-1)^{2n-i} \binom{n}{i} u^{i-1} (u')^i \right) = 1,$$

so $u \in R^\times$. $\square$

<u>Lemma 5</u>: Thm 1 holds for $p = pr : B \times F \to B$.

<u>Pf</u>: By Lemma XXV.8, the map

$$\begin{array}{ccc} K^*(B) \otimes K^*(F) & \xrightarrow{*} & K^*(B \times F) \\ \Sigma_{i=1}^{k} b_i \otimes i^*(c_i) & \longmapsto & \Sigma_{i=1}^{k} p^*(b_i) q^* i^*(c_i) \end{array}$$

is an isomorphism. Here $q : B \times F \to F$ is the projection. Thus $K^*(B \times F)$ is a free $K^*(B)$-module with basis

$$q^* i^*(c_1), \ldots, q^* i^*(c_k).$$

To prove the claim, it suffices to show that the $K^*(B)$-linear endomorphism of $K^*(B \times F)$ sending $q^* i^*(c_i)$ to c_i for all $i = 1, \ldots, k$ is an isomorphism. Let A be the matrix of this endomorphism (w.r.t. the basis $q^* i^*(c_1), \ldots, q^* i^*(c_k)$). Since all the c_i's have degree 0, the coefficients of A are in $K(B)$. By Lemmas 3 and 4, it is enough to show that πA is invertible, where

$$\pi : K(B) \longrightarrow H^0(B)$$

is the ring epimorphism. Notice that the units in $H^0(B)$ are precisely those locally constant functions $B \to \mathbb{Z}$ taking values in $\mathbb{Z}^\times = \{\pm 1\}$, so it is enough to show that $\det(\pi A) \in H^0(B)$ is such an element.

For $b \in B$, let $i_b : b \times F \hookrightarrow B \times F$ and $j_b : b \hookrightarrow B$ be the inclusions. Under the isomorphisms

$$K^*(B) \otimes K^*(F) \xrightarrow[\approx]{*} K^*(B \times F)$$

$$K^*(b) \otimes K^*(F) \xrightarrow[\approx]{*} K^*(b \times F),$$

the map $i_b^* : K^*(B\times F) \longrightarrow K^*(b \times F)$ corresponds to $j_b^* \otimes 1 : K^*(B) \otimes K^*(F) \longrightarrow K^*(b) \otimes K^*(F)$, so the diagram

$$\begin{tikzcd}
q^* i^*(c_i) \arrow[d, mapsto] & K^*(B\times F) \arrow[r, "i_b^*"] \arrow[d, "A"'] & K^*(b \times F) \arrow[d, "j_b^* A"] \\
c_i & K^*(B \times F) \arrow[r, "i_b^*"] & K^*(b \times F)
\end{tikzcd}$$

commutes. We conclude that $j_b^* A$ is the matrix for the endomorphism of $K^*(b \times F)$ sending $i^*(c_i) = i_b^* q^* i^*(c_i)$ to $i_b^*(c_i)$ for $i = 1, \ldots, k$. Since $i_b^*(c_1), \ldots, i_b^*(c_k)$ is a basis for $K(b \times F)$, it follows that $j_b^* A$ is invertible. Now observe that j_b^* factors as

$$\begin{tikzcd}
K(B) \arrow[r, "\pi"] \arrow[rrr, bend right, "j_b^*"'] & H^0(B) \arrow[r, "ev_b"] & \mathbb{Z} \arrow[r, "\Theta", "\approx"'] & K(b)
\end{tikzcd}$$

where $ev_b : H^0(B) \longrightarrow \mathbb{Z}$ is evaluation at b and Θ is the unique ring isomorphism $\mathbb{Z} \xrightarrow{\approx} K(b)$.

Thus

$$\begin{aligned} ev_b \det(\pi A) &= \det(ev_b \pi A) \\ &= \det(\theta^{-1} j_b^* A) \\ &= \theta^{-1} \det(j_b^* A) \\ &\in \theta^{-1} K(b)^\times \\ &= \mathbb{Z}^\times, \end{aligned}$$

and the claim follows. $\square$

Prop 6: Let (X, A) be a compact pair. Then the maps in the long exact sequence of (X, A)

$$\begin{tikzcd} K^*(X,A) \arrow[r, "j^*"] & K^*(X) \arrow[r, "i^*"] & K^*(A) \arrow[ll, bend left, "\delta"] \end{tikzcd}$$

are right $K^*(X)$-linear. (Here $i: A \hookrightarrow X$, $j: X \hookrightarrow (X, A)$ are the inclusions and $K^*(X)$ acts on $K^*(A)$ from the right by $a \cdot x = a\, i^*(x)$ and on $K^*(X,A)$ by

$$K^*(X,A) \otimes K^*(X) \xrightarrow{*} K^*((X,A)\times X) \xrightarrow{\Delta^*} K^*(X,A).)$$

Pf: The only difficulty is to show that δ is $K^*(X)$-linear. This follows if we can show that the diagram

$$\begin{tikzcd} K^{-n-1}(A) \otimes K^{-m}(X) \arrow[r, "\delta\otimes 1"] \arrow[d, "*"'] & K^{-n}(X,A) \otimes K^{-m}(X) \arrow[d, "*"] \\ K^{-n-m-1}(A \times X) \arrow[r, "\delta"] \arrow[d, "\Delta^*"'] & K^{-n-m}((X,A)\times X) \arrow[d, "\Delta^*"] \\ K^{-n-m-1}(A) \arrow[r, "\delta"] & K^{-n-m}(X,A) \end{tikzcd}$$

commutes for all $m, n \geq 0$. The bottom square commutes by naturality of δ, and the top square commutes by a lengthy but straightforward argument unrolling the definitions of $*$ and δ. As a part of the argument, one encounters the following commutative diagram:

$$\begin{tikzcd}
A_+ \wedge S^1 \wedge S^n \wedge X_+ \wedge S^m \arrow[r, dash, "\approx"'] \arrow[d, "\mathrm{perm.}"', "\approx"] & \Sigma A_+ \wedge S^n \wedge X_+ \wedge S^m \arrow[d, "\approx"', "\mathrm{perm.}"] & (X_+ \cup CA_+) \wedge S^n \wedge X_+ \wedge S^m \arrow[l, "q\partial(i) \wedge 1"'] \arrow[r, "\pi(i) \wedge 1"] \arrow[d, "\approx"', "\mathrm{perm.}"] & X/A \wedge S^n \wedge X_+ \wedge S^m \arrow[d, "\approx"', "\mathrm{perm.}"] \\
A_+ \wedge S^1 \wedge X_+ \wedge S^n \wedge S^m \arrow[r, dash, "\approx"'] \arrow[d, "\mathrm{perm.}"', "\approx"] & \Sigma A_+ \wedge X_+ \wedge S^n \wedge S^m \arrow[d, "\approx"'] & (X_+ \cup CA_+) \wedge X_+ \wedge S^n \wedge S^m \arrow[l, "q\partial(i) \wedge 1"'] \arrow[r, "\pi(i) \wedge 1"] \arrow[d, "\approx"'] & X/A \wedge X_+ \wedge S^n \wedge S^m \arrow[d, "\approx"'] \\
A_+ \wedge X_+ \wedge S^1 \wedge S^n \wedge S^m \arrow[r, dash, "\approx"'] & \Sigma (A_+ \wedge X_+) \wedge S^n \wedge S^m & ((X_+ \wedge X_+) \cup C(A_+ \wedge X_+)) \wedge S^n \wedge S^m \arrow[l, "q\partial(i \times 1) \wedge 1"'] \arrow[r, "\pi(i \times 1) \wedge 1"] & X_+ \wedge X_+ / A_+ \wedge X_+ \wedge S^n \wedge S^m
\end{tikzcd}$$

□

<u>Proof of Thm 1:</u> For a closed subspace $U \subset B$, write $E_U = p^{-1}(U)$. For $V \subset U \subset B$ closed subspaces, we consider the map

$$\begin{aligned}
\Phi_{(U,V)} : K^*(U,V) \otimes K^*(F) &\longrightarrow K^*(E_U, E_V) \otimes K^*(E_U) \longrightarrow K^*(E_U, E_V) \\
\textstyle\sum_i b_i \otimes i^*(c_i) &\longmapsto \textstyle\sum_i p^*(b_i) \otimes c_i | U
\end{aligned}$$

where $c_i | U$ denotes the restriction of $c_i \in K^*(E)$ to $K^*(E_U)$. Write $\Phi_U = \Phi_{(U,\emptyset)}$. We have the diagram

$$(**) \qquad \begin{tikzcd}
\cdots \arrow[r] & K^*(U,V) \otimes K^*(F) \arrow[r] \arrow[d, "\Phi_{(U,V)}"'] & K^*(U) \otimes K^*(F) \arrow[r] \arrow[d, "\Phi_U"] & K^*(V) \otimes K^*(F) \arrow[r] \arrow[d, "\Phi_V"] & \cdots \\
\cdots \arrow[r] & K^*(E_U, E_V) \arrow[r] & K^*(E_U) \arrow[r] & K^*(E_V) \arrow[r] & \cdots
\end{tikzcd}$$

The bottom row is the long exact sequence of the pair (E_U, E_V), and the top row is the long exact sequence of the pair (U,V) tensored with $K^*(F)$. Since $K^*(F)$ is free, the top row is exact (it is just a direct sum of the original sequences).

The diagram commutes, since we can interpolate between the two rows by

$$\cdots \to K^*(E_U, E_V) \otimes K^*(E_U) \to K^*(E_U) \otimes K^*(E_U) \to K^*(E_V) \otimes K^*(E_U) \to \cdots$$

The top part in the resulting diagram commutes by the naturality of the long exact sequence of a pair, and the bottom part commutes by Prop. 6.

Our task is to show that Φ_B is an isomorphism. Call a closed $U \subset B$ <u>good</u> if Φ_V is an isomorphism for every closed $V \subset U$. By Lemma 5 and compactness of B, we can cover B by a finite number of good closed subsets. Thus it is enough to show that if U_1 and U_2 are good, then so is $U_1 \cup U_2$. Let $V \subset U_1 \cup U_2$ be closed, and write $V_i = V \cap U_i$. Diagram (**) with $(V_1, V_1 \cap V_2)$ in place of (U,V) and the Five Lemma allow us to deduce that $\Phi_{(V_1, V_1 \cap V_2)}$ is an isomorphism. We have the commutative square

$$\begin{array}{ccc}
K^*(V, V_2) \otimes K^*(F) & \longrightarrow & K^*(V_1, V_1 \cap V_2) \otimes K^*(F) \\
\Phi_{(V,V_2)} \big\downarrow & & \approx \big\downarrow \Phi_{(V_1, V_1 \cap V_2)} \\
K^*(E_V, E_{V_2}) & \longrightarrow & K^*(E_{V_1}, E_{V_1 \cap V_2})
\end{array}$$

where the horizontal morphisms are induced by inclusions. Since these inclusions induce homeomorphisms $V_1/V_1 \cap V_2 \xrightarrow{\approx} V/V_2$ and $E_{V_1}/E_{V_1 \cap V_2} \xrightarrow{\approx} E_V/E_{V_2}$, the horizontal morphisms are isomorphisms. It follows that $\Phi(V, V_2)$ is an isomorphism. Now diagram (**) with (V, V_2) in place of $(\mathcal{V}, V)$ and the Five lemma imply that Φ_V is an isomorphism. Thus $\mathcal{V}_1 \cup \mathcal{V}_2$ is good, and the claim follows. $\square$