The splitting principle (cont.)

Recall: We want to prove

Thm 1 (Splitting principle): Sp. X is a compact Hausdorff space and let $\xi \to X$ be a complex v.b. Then there exist a compact Hausdorff space Y and a map $p: Y \to X$ s.t.

$$p^*: K^*(X) \longrightarrow K^*(Y)$$

is injective and $p^*\xi \longrightarrow Y$ splits as a direct sum of line bundles.

Last time, we proved

Thm 2 (Leray–Hirsch): Let $p: E \to B$ be a fibre bundle with E, B compact Hausdorff and fibre F a finite cell complex with only even-dimensional cells. Suppose there exist classes $c_1, \dots, c_k \in K(E)$ which restrict to a basis for $K(E_b)$ for each fibre $E_b = p^{-1}(b)$ of $p: E \to B$. Then $K^*(E)$ is a free $K^*(B)$-module with basis $\{c_1, \dots, c_k\}$. □

The main remaining ingredient in the proof of Thm 1 is the computation of $K(\mathbb{C}P^n)$.

Recall: $\mathbb{C}P^n$ is a finite cell complex with a single cell of dimension $2k$ for all $0 \le k \le n$. Thus $K^1(\mathbb{C}P^n) = 0$ and $K^0(\mathbb{C}P^n)$ is free abelian of rank $n+1$ (Cor. XXII.2).

<u>Thm 3</u>: $K(\mathbb{C}P^n) = \mathbb{Z}[L]/(L-1)^{n+1}$ where $L \to \mathbb{C}P^n$ is the dual of the canonical line bundle $\gamma^1 \to \mathbb{C}P^n$.

Thus in particular $K(\mathbb{C}P^n)$ has additive basis $1, L, \ldots, L^n$.

<u>Pf of Thm 3</u>: By induction on n. $\mathbb{C}P^0 = pt$, so the claim holds for $n=0$. For $n>0$, the long exact sequence of $(\mathbb{C}P^n, \mathbb{C}P^{n-1})$ gives a short exact sequence

$$0 \to \underset{\substack{\| \\ \tilde{K}(S^{2n}) \\ \wr\| \\ \mathbb{Z}}}{K(\mathbb{C}P^n, \mathbb{C}P^{n-1})} \longrightarrow K(\mathbb{C}P^n) \xrightarrow{\rho} K(\mathbb{C}P^{n-1}) \to 0.$$

The claim will follow if we can show

(A_n) $\quad \operatorname{Ker}(\rho)$ is generated by $(L-1)^n$.

Then induction and the above sequence imply that $1, L-1, \ldots, (L-1)^n$ is an additive basis for $K(\mathbb{C}P^n)$. Moreover, by (A_{n+1}) we have $(L-1)^{n+1} = 0$ in $K(\mathbb{C}P^n)$, so $K(\mathbb{C}P^n) = \mathbb{Z}[L]/(L-1)^{n+1}$ as claimed.

It remains to show (A_n) for all $n>0$. Notice that $K(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \approx \tilde{K}(S^{2n})$. We know that $\tilde{K}(S^{2n})$ is generated by $(H-1)^{*n}$. The basic idea is to relate the internal product

$$K(\mathbb{C}P^n) \otimes \cdots \otimes K(\mathbb{C}P^n) \longrightarrow K(\mathbb{C}P^n)$$

to the external product

$$\tilde{K}(S^2) \otimes \cdots \otimes \tilde{K}(S^2) \xrightarrow{*} \tilde{K}(S^{2n}).$$

Instead of interpreting $\mathbb{C}P^n$ as the orbit space S^{2n+1}/S^1 where $S^1 \subset \mathbb{C}$ acts on $S^{2n+1} \subset \mathbb{C}^{n+1}$ by scalar multiplication, we may interpret $\mathbb{C}P^n$ as the orbit space

$$\mathbb{C}P^n = \partial(D_1^2 \times \cdots \times D_{n+1}^2)/S^1$$

where D_i^2 is the unit disk in the i-th coordinate in $\mathbb{C}^{n+1}$, and the action is again by scalar multiplication. We have a decomposition

$$\partial(D_1^2 \times \cdots \times D_{n+1}^2) = \bigcup_{i=1}^{n+1} D_1^2 \times \cdots \times \partial D_i^2 \times \cdots \times D_{n+1}^2$$

into S^1-invariant subspaces. The quotient of the i-th member of the union by the S^1-action is a subspace $C_i \subset \mathbb{C}P^n$ homeomorphic to $D_1^2 \times \cdots \times D_{n+1}^2$ with D_i^2 omitted. Explicitly, a homeomorphism is given by

$$\varphi_i : D_1^2 \times \cdots \times \widehat{D_i^2} \times \cdots \times D_{n+1}^2 \xrightarrow{\approx} C_i$$

$$(z_1, \ldots, \hat{z}_i, \ldots, z_{n+1}) \longmapsto [z_1 : \cdots : \underset{i}{1} : \cdots z_{n+1}]$$

where $\widehat{\ }$ indicates omission. Thus we have a decomposition

$$\mathbb{C}P^n = \bigcup_{i=1}^{n+1} C_i$$

with each C_i homeomorphic to D^{2n}.

Identify C_{n+1} and $D_1^2 \times \cdots \times D_n^2$ via φ_{n+1},
and let
$$\partial_i C_{n+1} = D_1^2 \times \cdots \times \partial D_i^2 \times \cdots \times D_n^2 .$$

We now have the following commutative diagram (where the morphisms are induced by inclusions unless indicated otherwise):

$$(*)\quad \begin{tikzcd}
K(D_1^2,\partial D_1^2)\otimes\cdots\otimes K(D_n^2,\partial D_n^2) \arrow[r, "*", "\approx\ \text{②}"'] & K(D_1^2\times\cdots\times D_n^2,\partial(D_1^2\times\cdots\times D_n^2)) & \\
K(C_{n+1},\partial_1 C_{n+1})\otimes\cdots\otimes K(C_{n+1},\partial_n C_{n+1}) \arrow[u, "\text{①}\ \approx"] \arrow[r] & K(C_{n+1},\partial C_{n+1}) \arrow[u, "\text{③}\ \approx"', "\varphi_{n+1}^*"] & \\
K(\mathbb{C}P^n, C_1)\otimes\cdots\otimes K(\mathbb{C}P^n, C_n) \arrow[u] \arrow[d] \arrow[r] & K(\mathbb{C}P^n, C_1\cup\cdots\cup C_n) \arrow[u, "\text{④}\ \approx"] \arrow[d] \arrow[dr, "\approx\text{⑤}"] & \\
K(\mathbb{C}P^n)\otimes\cdots\otimes K(\mathbb{C}P^n) \arrow[r] & K(\mathbb{C}P^n) & K(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \arrow[l]
\end{tikzcd}$$

Several of the maps in the diagram are isomorphisms. This can be justified as follows:

① : $(D_i^2,\partial D_i^2) \hookrightarrow (C_{n+1},\partial_i C_{n+1}) = D_1^2\times\cdots\times(D_i^2,\partial D_i^2)\times\cdots\times D_n^2$ is a homotopy equivalence of pairs (with homotopy inverse given by the projection).

② : By comparison with $\tilde{K}(S^2)\otimes\cdots\otimes\tilde{K}(S^2) \xrightarrow[\approx]{*} \tilde{K}(S^{2n})$.

③ : φ_{n+1} is a homeomorphism.

④ $C_{n+1} \hookrightarrow \mathbb{C}P^n$ induces a homeomorphism
$$C_{n+1}/\partial C_{n+1} \xrightarrow{\approx} \mathbb{C}P^n / C_1\cup\cdots\cup C_n .$$

(5): Notice that $\mathbb{C}P^{n-1} = \{[z_1 : \cdots : z_n : 0] \in \mathbb{C}P^{n+1}\}$ does not intersect C_{n+1}, so $\mathbb{C}P^{n-1} \subset C_1 \cup \cdots \cup C_n$. It is easy to show that the quotient map

$$\mathbb{C}P^n/\mathbb{C}P^{n-1} \longrightarrow \mathbb{C}P^n/C_1 \cup \cdots \cup C_n$$

is a pointed homotopy equivalence. (We have a commutative square

$$\begin{tikzcd}
{[v : 1-|v|]} & \mathbb{C}P^n/\mathbb{C}P^{n-1} \arrow[r] & \mathbb{C}P^n/C_1 \cup \cdots \cup C_n & {[z_1 : \cdots : z_n : 1]} \\
{[v]} \arrow[u, mapsto] & D^{2n}/\partial D^{2n} \arrow[u, "\approx"] \arrow[r, "\alpha"] & D_1^2 \times \cdots \times D_n^2 / \partial(D_1^2 \times \cdots \times D_n^2) \arrow[u] & {[(z_1, \ldots, z_n)]} \arrow[u, mapsto]
\end{tikzcd}$$

where α is induced by

$$D^{2n} \longrightarrow D_1^2 \times \cdots \times D_n^2$$

$$v \longmapsto \begin{cases} v/1-|v| & \text{if } \|v\|_\infty \leq 1-|v| \\ v/\|v\|_\infty & \text{otw} \end{cases}$$

($\|(z_1, \ldots, z_n)\|_\infty = \max_{1 \leq i \leq n} |z_i|$). Now check that

$$D_1^2 \times \cdots \times D_n^2 \longrightarrow D^{2n}$$

$$v \longmapsto \frac{\|v\|_\infty}{|v|} v$$

induces a homotopy inverse for α.)

Since C_i is contractible, the map

$$K(\mathbb{C}P^n, C_i) \xrightarrow{(\mathrm{incl})^*} K(\mathbb{C}P^n)$$

gives an isomorphism

$$K(\mathbb{C}P^n, C_i) \xrightarrow{\approx} \tilde{K}(\mathbb{C}P^n).$$

It follows that $L-1 \in \tilde{K}(\mathbb{C}P^n)$ lifts to a unique class $x_i \in K(\mathbb{C}P^n, C_i)$. Interpret

$$\mathbb{C}P^1 = D^2 \cup D^2_\infty$$

where

$$D^2 = \{ [z:1] \mid |z| \le 1 \}$$
$$D^2_\infty = \{ [1:z] \mid |z| \le 1 \}.$$

Now the commutative diagram

$$\begin{tikzcd}
{[z_1:z_2]} \arrow[d, mapsto] & \mathbb{C}P^1 \arrow[r, hook] \arrow[d] & (\mathbb{C}P^1, D^2_\infty) \arrow[dr] & (D^2, \partial D^2) \arrow[l, hook', "{[z:1] \leftarrow\!\shortmid z}"'] \arrow[d] & z \arrow[d, mapsto] \\
{[0:\cdots:\underset{i}{z_1}:\cdots 0:z_2]} & \mathbb{C}P^n \arrow[rr, hook] & & (\mathbb{C}P^n, C_i) & {[0:\cdots:\underset{i}{z}:\cdots:0:1]}
\end{tikzcd}$$

gives a commutative diagram

$$\begin{tikzcd}
H-1 & \tilde{K}(\mathbb{C}P^1) & K(\mathbb{C}P^1, D^2_\infty) \arrow[l, "\approx"'] \arrow[r, "\approx", "\text{excision}"'] & K(D^2, \partial D^2) \\
 & \tilde{K}(\mathbb{C}P^n) \arrow[u] & & K(\mathbb{C}P^n, C_i) \arrow[u] \arrow[ul] \arrow[ll, "\approx"'] \\
L-1 \arrow[uu, mapsto] & & & x_i \arrow[lll, mapsto]
\end{tikzcd}$$

We deduce that in $(*)$, the element $x_i \in K(\mathbb{C}P^n, C_i)$ maps to the generator in $K(D^2_i, \partial D^2_i)$. (A_n) now follows from diagram $(*)$. Notice that $\ker(\beta)$ is the image of the map

$$K(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \longrightarrow K(\mathbb{C}P^n). \qquad \square$$

Recall that if $\xi \to X$ is a complex v.b. of constant dimension n, we have a fibre bundle $P(\xi) \to X$ with fibre $P(\xi)_x$ over $x \in X$ the space $P(\xi_x) \approx \mathbb{C}P^{n-1}$ of lines in ξ_x. Over $P(\xi)$, we have a canonical line bundle

$$L^* = \{ (\ell, v) \in P(\xi) \times \xi \mid v \in \ell \} \longrightarrow P(\xi)$$
$$(\ell, v) \longmapsto \ell$$

with fibre $L^*_\ell \approx \ell$. Write L for the dual of L^*. By Thm 3, the elements

$$1, L, \dots, L^{n-1} \in K(P(\xi))$$

restrict to a basis of $K(P(\xi)_x)$ for each $x \in X$, so by Thm 2, $K^*(P(\xi))$ is a free $K^*(X)$-module with basis $\{1, L, \dots, L^{n-1}\}$.

<u>Proof of Thm 1</u>: Consider first the case where ξ is of constant dimension n. If $n = 0$, we may take $Y = X$, $p = id_X$. Sp. $n > 0$. By the above discussion, since $1 \in K(P(\xi))$ is among the basis elements, the map

$$K(X) \xrightarrow{\pi^*} K(P(\xi))$$

induced by the projection $\pi : P(\xi) \to X$ is a monomorphism. Moreover, the pullback $\pi^*(\xi)$ contains the canonical line bundle L^* as a subbundle; the embedding $L^* \to \pi^*(\xi)$ is induced by

$$\begin{array}{ccc} (\ell, v) & \longmapsto & v \\ L^* & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ P(\xi) & \xrightarrow{\pi} & X \end{array}$$

Thus $\pi^*(\xi)$ splits as a sum

$$\pi^*(\xi) \approx L^* \oplus \xi'$$

for some $(n-1)$-dimensional v.b. $\xi' \to P(\xi)$. The claim now follows by induction. In the general case, X splits as a disjoint union

$$X = X_1 \amalg \cdots \amalg X_k$$

of subspaces s.t. $\xi | X_i$ has constant dimension for each $i = 1, \ldots, k$. The claim follows by considering each $\xi | X_i$ separately. $\square$

Let $\xi \to X$ be an n-dimensional complex v.b. We saw earlier that $K^*(P(\xi))$ is a free $K^*(X)$-module with basis $\{1, L, \ldots, L^{n-1}\}$. Let us now determine the multiplicative structure of $K^*(P(\xi))$. To do so, it is enough to express L^n in terms of $1, L, \ldots, L^{n-1}$. The pullback of ξ over $P(\xi)$ splits as a sum $L^* \oplus \xi'$ with ξ' $(n-1)$-dimensional. Continuing to write ξ for the pullback, we have

$$\lambda_t(\xi) = \lambda_t(L^*)\lambda_t(\xi')$$

so that

$$\lambda_t(\xi') = \lambda_t(\xi)\lambda_t(L^*)^{-1} = \lambda_t(\xi)(1 + L^* t)^{-1}$$
$$= \lambda_t(\xi)\left(\sum_{i=0}^{\infty} (-1)^i (L^*)^i t^i\right).$$

Since ξ' is $(n-1)$-dimensional, $\lambda^n(\xi') = 0$. Comparing the coefficients of t^n, we get that

$$0 = \sum_{i=0}^{n} (-1)^i \lambda^i(\xi)(L^*)^{n-i}$$
$$= \sum_{i=0}^{n} (-1) \lambda^i(\xi) L^{i-n}$$

in $K(P(\xi))$. Multiplying by L^n now gives the identity

$$\sum_{i=0}^{n} (-1)^i \lambda^i(\xi) L^i = 0.$$

Notice that the coefficient $(-1)^n \lambda^n(\xi)$ of L^n on the left hand side is a unit, so this allows us to write L^n as a linear combination of lower powers of L. We conclude that

Prop 4: For $\xi \to X$ an n-dimensional complex v.b., the ring $K^*(P(\xi))$ is isomorphic to

$$K^*(X)[L] \Big/ \left(\sum_{i=0}^{n} (-1)^i \lambda^i(\xi) L^i\right). \quad \square$$