STRING TOPOLOGY AND TWISTED K-THEORY

A DISSERTATION
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE COMMITTEE ON GRADUATE STUDIES
OF STANFORD UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

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August 2010

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

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Acknowledgements

I would like to thank my advisor Ralph Cohen for his advice and guidance during my time at Stanford. He has always been patient and supportive, and he will be the model I will be looking up to when the time comes for me to advise students of my own. In addition, I would like to thank Professor Sören Illman for guiding me during my Helsinki years. It was in part his example that gave me the courage and vision to pursue my interest in topology abroad.

At Stanford, I have benefited from mathematical discussions with a number of people, and I would like to mention Søren Galatius and Andrew Blumberg in particular. I am grateful to them for sharing their knowledge and insights. Also, I am grateful to my fellow students for their company and camaraderie that has enriched my life at Stanford. Zach, Tracy, Josh, Nikola, David, Jack, Penka, Eric, Robin, Jon, and especially Ben, thank you all for making my time at Stanford a much more enjoyable experience.

Finally, I would like to thank my family. Even while I have been living far away, you have remained an anchor in my life.

During my studies, I have been partially supported by the Finnish Cultural Foundation. It is my pleasure to acknowledge their generous support.

Contents

Chapter 1

Introduction

The motivation for the mathematics in this thesis comes from the desire to connect twisted equivariant K-groups $K_G^\tau(G^{\mathrm{ad}})$ for a compact Lie group G to the string topology of the classifying space BG. (Here G^{ad} denotes the group G equipped with the conjugation action.) An important result of Freed, Hopkins and Teleman [FHT07b, FHT05, FHT03] relates these K-groups to the representation theory of the loop group LG, including Verlinde algebras, making these K-groups very interesting objects to study and the prospect of connecting them to string topology a tantalizing one. As the Borel construction of G^{ad} is homotopy equivalent to the free loop space LBG, it does not seem far-fetched to expect that there exists such a connection.

The thesis has two parts. In the first part, consisting of Chapter 2, we prove a generalization of the Atiyah–Segal completion theorem [AS69] to twisted K-theory. This is our Theorem 2. In an alternative formulation, the theorem provides a connection between the twisted equivariant K-theory of a G-space X and the non-equivariant K-theory of the Borel construction of X, and in this guise the theorem has been successfully used by Kriz, Westerland and Levin [KWL09] to connect the K-groups $K_G^\tau(G^{\mathrm{ad}})$ to the Gruher–Salvatore string topology of BG [GS08]. Of course, Theorem 2 is also of interest independent of this application: the untwisted Atiyah–Segal completion theorem is one of the basic theorems in classical equivariant K-theory, and in the twisted setting the twisted version should play a similarly fundamental role.

The backdrop for the second part of the thesis is formed by two different families of two-dimensional field theories: first, a family of field theories constructed by Freed, Hopkins and Teleman [FHT07a] sending the circle to the groups $K^{\tau}_{G}(G^{\mathrm{ad}})$; and second, the Chataur–Menichi [CM07] string topology of BG, a family of field theories sending the circle to the homology groups $H_*(LBG; k)$, where k is a field. In both cases the homotopy theory underlying the field theories is similar, and what we would like to do is to make this similarity explicit by presenting a single construction that comes as close as possible to capturing both kinds of field theories. We can state our motivating conjecture as follows.

Conjecture 1. *Let R be a commutative S-algebra. Then, given a piece of orientation data we call a universal R-orientation (see Definition 37), there exists a non-unital, non-counital Homological Conformal Field Theory taking values in the homotopy category of R-modules and sending the circle to an R-module whose homotopy groups are the twisted R-homology groups of LBG, with the twisting determined by the universal R-orientation. Taking R to be the Eilenberg–Mac Lane spectrum Hk for a field k, a suitable choice of universal Hk-orientation gives rise to Chataur and Menichi's string topology of BG, while taking R to be the complex K-theory spectrum K, we obtain field theories related to the Freed–Hopkins–Teleman field theories.*

In the second part of the thesis, we present work towards the construction of the field theories of Conjecture 1. This is explicitly our subject in Chapter 5, where our main result, Theorem 41, asserts the existence of a field-theory operation associated with a single cobordism W. This operation arises from the correspondence diagram

$$\mathrm{map}(\partial_0 W, BG) \leftarrow \mathrm{map}(W, BG) \rightarrow \mathrm{map}(\partial_1 W, BG)$$

by making use of a pretransfer-type umkehr map induced by the first arrow. The construction of such umkehr maps in the twisted setting is done in Chapter 4, where the main result is Theorem 26, a slight partial generalization (from fiber bundles to fibrations) of the Fiberwise Costenoble–Waner Duality Theorem of May and Sigurdsson ([MS06], Theorem 19.5.2). Our discussion in Chapters 4 and 5 draws very heavily from parametrized homotopy theory, so in Chapter 3 we offer a review of parametrized homotopy theory, mainly relying on May and Sigurdsson's monograph [MS06]. While there are no new results in the chapter, we hope its inclusion will help the reader previously unacquainted with parametrized homotopy theory to navigate the exposition in Chapters 4 and 5.

Chapter 2

The Atiyah–Segal completion theorem in twisted K-theory

2.1 Introduction

The aim of this chapter is to prove the following twisted analogue of the Atiyah–Segal completion theorem [AS69].

Theorem 2. *Let X be a finite G-CW complex, where G is a compact Lie group. Then the projection $\pi : EG \times X \to X$ induces an isomorphism*

$$K_G^{\tau+*}(X)^\wedge_{I_G} \xrightarrow{\approx} K_G^{\pi^*(\tau)+*}(EG \times X)$$

for any twisting τ corresponding to an element of $H^1_G(X; \mathbf{Z}/2) \oplus H^3_G(X; \mathbf{Z})$.

Here $I_G \subset R(G)$ is the augmentation ideal of the representation ring $R(G)$ and $(-)^\wedge_{I_G}$ indicates completion. The classical theorem is the case $\tau = 0$. Theorem 2 generalizes a result by C. Dwyer, who has proved the theorem in the case where G is finite and the twisting τ is given by a cocycle of G [Dwy]. While versions of the theorem for compact Lie groups have been known to experts (for example, such a theorem is used in the proof of [FHT08], Proposition 3.11), to our knowledge no proof of the general theorem appears in the current literature. Our goal is to fill in this gap.

We shall prove Theorem 2 in two stages. First we prove the theorem in the special case of a twisting arising from a graded central extension

$$1 \to \mathbb{T} \to G^\tau \to G \to 1, \qquad \epsilon : G \to \mathbf{Z}/2.$$

For such twistings, twisted G-equivariant K-groups correspond to certain direct summands of untwisted G^τ-equivariant K-groups, and the Adams–Haeberly–Jackowski–May argument contained in [AHJM88a] goes through with these summands to prove the theorem in this case. It follows that the theorem holds when X is a point, and the general theorem then follows by a Mayer–Vietoris argument.

As our definition of twisted K-theory, we use Freed, Hopkins and Teleman's elaboration [FHT07b] of the Atiyah–Segal model developed in [AS04]. Thus for a G-space X, the notation $K_G^{\tau+*}(X)$ is a shorthand for $K^{\tau+*}(X//G)$, where $X//G$ is the quotient groupoid of X. Of course, the completion theorem should remain true in any reasonable alternative model for twisted equivariant K-theory as well.

The chapter is structured as follows. In Section 2.2 we describe a pro-group valued variant of K-theory which we shall employ in Section 2.3 to handle the case of a twisting given by a central extension. Section 2.4 then contains a proof of the general theorem.

2.2 A convenient cohomology theory

We shall now describe a cohomology theory which will be used in the next section to prove the completion theorem for twistings arising from a graded central extension. Let

$$1 \to C \to \tilde{G} \to G \to 1$$

be a central extension of a compact Lie group G by a commutative compact Lie group C, and let X be a finite G-CW complex. Via the map $\tilde{G} \to G$ we can view X as a $\tilde{G}$-space on which C acts trivially. The semigroup $\mathrm{Vect}_{\tilde{G}}(X)$ of isomorphism classes of $\tilde{G}$-equivariant vector bundles over X decomposes as a direct sum

$$\mathrm{Vect}_{\tilde{G}}(X) = \bigoplus_{\pi \in \hat{C}} \mathrm{Vect}_{\tilde{G}}(X)(\pi) \tag{2.1}$$

where $\hat{C}$ denotes the set of isomorphism classes of irreducible representations of C, and where $\mathrm{Vect}_{\tilde{G}}(X)(\pi)$ is the semigroup of isomorphism classes of those $\tilde{G}$-vector bundles ξ over X with the property that the fibers of ξ are π-isotypical as representations of C, that is, isomorphic to sums of copies of π. The decomposition (2.1) leads to a decomposition

$$K^*_{\tilde{G}}(X) = \bigoplus_{\pi \in \hat{C}} K^*_{\tilde{G}}(X)(\pi) \tag{2.2}$$

and similarly for reduced K-groups.[1] Here $K^0_{\tilde{G}}(X)(\pi)$ is the Grothendieck group of $\mathrm{Vect}_{\tilde{G}}(X)(\pi)$, and $K^q_{\tilde{G}}(X)(\pi)$ for non-zero q is defined by using the suspension isomorphism and the Bott periodicity map. By inspection and definition, the decomposition (2.2) is compatible with G-equivariant maps of spaces, with the suspension isomorphism, with the Thom isomorphism for G-equivariant vector bundles, and, as a special case, with the Bott periodicity map. Thus for each $\pi \in \hat{C}$, we can view $K^*_{\tilde{G}}(-)(\pi)$ as a $\mathbf{Z}/2$-graded cohomology theory defined on finite G-CW complexes and taking values in graded $R(G)$-modules.

Although the decomposition (2.2) fails for infinite X in general, it is possible to extend each one of the theories $K^*_{\tilde{G}}(-)(\pi)$ to infinite G-CW complexes by means of suitable classifying spaces. However, since having the theories available for finite complexes suffices for most of our purposes, we will not elaborate this point. Instead, we point the reader to the proof of Proposition 3.5 in [FHT07b] for a description of the appropriate classifying space when π is the defining presentation of the circle group $\mathbb{T}$, which is the only case where we will need to apply $K^*_{\tilde{G}}(-)(\pi)$ to an infinite complex in the sequel.

Our interest in the groups $K^*_{\tilde{G}}(X)(\pi)$ is explained by the following proposition.

[1] In fact, tensor product makes $K^*_{\tilde{G}}(X)$ into a $\hat{C}$-graded algebra where the modules $K^*_{\tilde{G}}(X)(\pi)$ are the homogeneous parts. However, we shall not need this graded algebra structure.

Recall that a graded central extension of a group G is a central extension of G together with a homomorphism from G to $\mathbf{Z}/2$.

Proposition 3 (A reformulation of Proposition 3.5 of [FHT07b]). *Let G be a compact Lie group, let X be a G-space, and let τ be the twisting given by a graded central extension*

$$1 \to \mathbb{T} \to G^\tau \to G \to 1, \qquad \epsilon : G \to \mathbf{Z}/2$$

of G by the circle group $\mathbb{T}$. Let $S^1(\epsilon)$ denote the one-point compactification of the 1-dimensional representation of G given by $(-1)^\epsilon$. Then there is a natural isomorphism

$$K_G^{\tau+n}(X) \approx \tilde{K}_{G^\tau}^{n+1}(X_+ \wedge S^1(\epsilon))(1)$$

where "1" refers to the defining representation of $\mathbb{T}$. □

The groups $K^*_{\tilde{G}}(X)(\pi)$ are not what we are going to use in the next section. Instead, we need pro-group valued versions completed at the augmentation ideal. (For background material on pro-groups, we refer the reader to [AHJM88b].) Given an arbitrary G-CW complex X and an irreducible representation π of C, we let $\mathbf{K}^*_{\tilde{G}}(X)(\pi)$ denote the pro-$R(G)$-module

$$\mathbf{K}^*_{\tilde{G}}(X)(\pi) = \{K^*_{\tilde{G}}(X_\alpha)(\pi)\}_\alpha$$

where X_α runs over all finite G-CW subcomplexes of X and the structure maps of the pro-system are those induced by inclusions between subcomplexes. The groups of our interest are then given by the pro-$R(G)$-modules

$$\mathbf{K}^*_{\tilde{G}}(X)(\pi)^\wedge_{I_G} = \{K^*_{\tilde{G}}(X_\alpha)(\pi) \,/\, I_G^n \cdot K^*_{\tilde{G}}(X_\alpha)(\pi)\}_{\alpha,n},$$

where X_α again runs over the finite G-subcomplexes of X, n runs over the natural numbers, and the structure maps of the pro-system are the evident ones. We think of $\mathbf{K}^*_{\tilde{G}}(X)(\pi)^\wedge_{I_G}$ as the completion of $\mathbf{K}^*_{\tilde{G}}(X)(\pi)$ with respect to the augmentation ideal I_G. Reduced variants $\tilde{\mathbf{K}}^*_{\tilde{G}}(X)(\pi)$ and $\tilde{\mathbf{K}}^*_{\tilde{G}}(X)(\pi)^\wedge_{I_G}$ for a based G-CW complex X are defined in a similar way using the reduced groups $\tilde{K}^*_{\tilde{G}}(X_\alpha)(\pi)$, where X_α now runs through the finite G-CW subcomplexes of X containing the base point. The crucial feature of the groups $\mathbf{K}^*_{\tilde{G}}(X)(\pi)\widehat{_{I_G}}$ for us is that they form a cohomology theory on the category of G-CW complexes (and therefore, by G-CW approximation, on the category of all G-spaces). Phrased in terms of the reduced groups, this means that the following axioms hold.

1. (Homotopy invariance) If X and Y are based G-CW complexes and $f, g : X \to Y$ are homotopic through based G-equivariant maps, then the induced maps

$$f^*, g^* : \tilde{\mathbf{K}}^*_{\tilde{G}}(Y)(\pi)\widehat{_{I_G}} \to \tilde{\mathbf{K}}^*_{\tilde{G}}(X)(\pi)\widehat{_{I_G}}$$

are equal.

2. (Exactness) If X is a based G-CW complex and A is a subcomplex of X containing the base point, then the sequence

$$\tilde{\mathbf{K}}^*_{\tilde{G}}(X/A)(\pi)\widehat{_{I_G}} \to \tilde{\mathbf{K}}^*_{\tilde{G}}(X)(\pi)\widehat{_{I_G}} \to \tilde{\mathbf{K}}^*_{\tilde{G}}(A)(\pi)\widehat{_{I_G}}$$

is pro-exact.

3. (Suspension) For each q, there exists a natural isomorphism

$$\Sigma : \tilde{\mathbf{K}}^q_{\tilde{G}}(X)(\pi)\widehat{_{I_G}} \approx \tilde{\mathbf{K}}^{q+1}_{\tilde{G}}(\Sigma X)(\pi)\widehat{_{I_G}}$$

4. (Additivity) If X is the wedge sum of a family $\{X_i\}_{i\in I}$ of based G-CW complexes, the inclusions $X_i \hookrightarrow X$ induce an isomorphism

$$\tilde{\mathbf{K}}^*_{\tilde{G}}(X)(\pi)\widehat{_{I_G}} \xrightarrow{\approx} \prod_{i\in I} \tilde{\mathbf{K}}^*_{\tilde{G}}(X_i)(\pi)\widehat{_{I_G}}$$

The only difficulties in verifying these properties arise from the exactness axiom.

Proposition 4. *The functor* $\tilde{\mathbf{K}}^*_{\tilde{G}}(-)(\pi)\widehat{_{I_G}}$ *satisfies the exactness axiom.*

Sketch of proof. As in [AHJM88b], because the ring $R(G)$ is Noetherian (see [Seg68], Corollary 3.3), the result follows from the Artin–Rees lemma once $\tilde{K}^*_{\tilde{G}}(Z)(\pi)$ is known to be finitely generated as an $R(G)$-module for any finite based G-CW complex Z. We shall prove that $\tilde{K}^*_{\tilde{G}}(Z)(\pi)$ is finitely generated by reduction to successively simpler cases. Filtering Z by skeleta and using the wedge and suspension axioms, we see that it is enough to consider the case where $Z = G/H_+$ for some closed subgroup H of G. Let $\tilde{H}$ denote the inverse image of H in $\tilde{G}$. Then $\tilde{H}$ is a central extension of H by C, and we have $\tilde{G}$-equivariant isomorphisms

$$G/H \approx (\tilde{G}/C)/(\tilde{H}/C) \approx \tilde{G}/\tilde{H}.$$

The $R(G)$-module isomorphisms

$$K^*_{\tilde{G}}(G/H) \approx K^*_{\tilde{G}}(\tilde{G}/\tilde{H}) \approx K^*_{\tilde{H}}(\mathrm{pt})$$

preserve the direct sum decomposition (2.2), whence we obtain an isomorphism

$$K^*_{\tilde{G}}(G/H)(\pi) \approx K^*_{\tilde{H}}(\mathrm{pt})(\pi).$$

Here the latter group can be identified with the summand $R(\tilde{H})(\pi)$ of $R(\tilde{H})$ generated by those representations of $\tilde{H}$ which restrict to π-isotypical representations of C. The $R(G)$-module structure on $R(\tilde{H})(\pi)$ arises from its $R(H)$-module structure via the map $R(G) \to R(H)$, and since $R(H)$ is finite over $R(G)$ ([Seg68], Proposition 3.2), we are reduced to showing that $R(\tilde{H})(\pi)$ is finite as an $R(H)$-module.

Now consider the restriction

$$R(\tilde{H}) \to \prod_S R(S) \tag{2.3}$$

where the S runs through the conjugacy classes of Cartan subgroups of $\tilde{H}$ (conjugacy classes of such subgroups are finite in number and each one of the subgroups is closed, Abelian and contains the central subgroup C). This map is injective ([Seg68], Proposition 1.2), whence $R(\tilde{H})(\pi)$ is a subgroup of $\prod_S R(S)(\pi)$. Therefore it is enough to show that $R(S)(\pi)$ is finite as an $R(H)$-module for each S. The $R(H)$-module structure on $R(S)(\pi)$ arises from its structure of an $R(S/C)$-module via the map of representation rings induced by the inclusion $S/C \hookrightarrow H$, and as $R(S/C)$ is finite over $R(H)$, it is enough to prove that that $R(S)(\pi)$ is finite over $R(S/C)$.

We shall now show that $R(S)(\pi)$ is in fact a free $R(S/C)$-module with one generator. To prove this, recall that for a compact Abelian Lie group A, tensor product gives the set of irreducible representations $\hat{A}$ the structure of a finitely generated Abelian group, and that the representation ring of A is given by the group ring $\mathbf{Z}[\hat{A}]$. Moreover, our exact sequence of compact Abelian groups

$$1 \to C \to S \to S/C \to 1$$

gives rise to an exact sequence

$$1 \to \widehat{S/C} \to \hat{S} \to \hat{C} \to 1.$$

From this it is clear that the summand $R(S)(\pi)$ of $R(S) = \mathbf{Z}[\hat{S}]$ is the subgroup freely generated by members of the coset of $\widehat{S/C}$ in $\hat{S}$ mapping to π in $\hat{C}$, with the $R(S/C)$-module structure arising from the action of $\widehat{S/C}$ on the coset. Thus any representative of the coset will form an $R(S/C)$-basis for $R(S)(\pi)$, and we are done. □

The following two lemmas point out further useful properties of the theories $\mathbf{K}^*_{\tilde{G}}(-)(\pi)\widehat{_{I_G}}$.

Lemma 5. *Let H be a closed subgroup of G, and let X be a based H-CW complex. Then there is a natural isomorphism of pro-$R(G)$-modules*

$$\tilde{\mathbf{K}}^*_{\tilde{G}}(G_+ \wedge_H X)(\pi)\widehat{_{I_G}} \approx \tilde{\mathbf{K}}^*_{\tilde{H}}(X)(\pi)\widehat{_{I_H}},$$

where $\tilde{H}$ denotes the inverse image of H in $\tilde{G}$.

Proof. Observe that the H-CW structure on X gives rise to a G-CW structure on $G_+ \wedge_H X$, and that as X_α runs over the finite H-CW subcomplexes of X, $G_+ \wedge_H X_\alpha$

runs over the finite G-CW subcomplexes of $G_+ \wedge_H X$. Now the lemma follows from the $\tilde{G}$-equivariant isomorphism

$$\tilde{G} \wedge_{\tilde{H}} X_\alpha \approx G \wedge_H X_\alpha;$$

from the change of groups isomorphism

$$\tilde{K}^*_{\tilde{G}}(\tilde{G} \wedge_{\tilde{H}} X_\alpha) \approx \tilde{K}^*_{\tilde{H}}(X_\alpha);$$

from the compatibility of this isomorphism with the decomposition (2.2); and from the fact that the I_G-adic and I_H-adic topologies on an $R(H)$-module coincide (see [Seg68], Corollary 3.9). □

Lemma 6. *Let X be a free G-CW complex. Then there is a natural isomorphism* $\mathbf{K}^*_{\tilde{G}}(X)(\pi)\widehat{_{I_G}} \approx \mathbf{K}^*_{\tilde{G}}(X)(\pi)$.

Proof. (Compare with the proof of Proposition 4.3 in [AS69].) Let X_α be a finite G-CW subcomplex of X. Since the action of G on X is free, we have an isomorphism

$$K_G(X_\alpha) \xrightarrow{\approx} K(X_\alpha/G).$$

Pick a base point for X_α/G. Then the diagram

$$\begin{array}{ccccc} R(G) & \longrightarrow & K_G(X_\alpha) & \xrightarrow{\approx} & K(X_\alpha/G) \\ \downarrow & & & & \downarrow \\ \mathbf{Z} & = & = & = & \mathbf{Z} \end{array}$$

commutes, whence the composite of the maps in the top row sends I_G into $\tilde{K}(X_\alpha/G)$. However, since X_α/G is a finite CW-complex, the elements of $\tilde{K}(X_\alpha/G)$ are nilpotent. Because $R(G)$ is Noetherian, the ideal I_G is finitely generated, and it follows that for large enough n, the image of I^n_G in $K_G(X_\alpha)$ vanishes. Thus

$$I^n_G \cdot K^*_{\tilde{G}}(X_\alpha)(\pi) = 0$$

for large n, and therefore

$$\begin{aligned}\mathbf{K}^*_{\tilde{G}}(X)(\pi)^{\widehat{}}_{I_G} &= \{K^*_{\tilde{G}}(X_\alpha)(\pi)/I^n_G \cdot K^*_{\tilde{G}}(X_\alpha)(\pi)\}_{\alpha,n} \\ &\approx \{K^*_{\tilde{G}}(X_\alpha)(\pi)\}_\alpha \\ &= \mathbf{K}^*_{\tilde{G}}(X)(\pi)\end{aligned}$$

as claimed. □

Remark 7. The main technical benefit of introducing the pro-group-valued theories $\mathbf{K}^*_{\tilde{G}}(-)(\pi)$ and $\mathbf{K}^*_{\tilde{G}}(-)(\pi)^{\widehat{}}_{I_G}$ is that they allow us to sidestep problems with exactness that would otherwise complicate the proof. The source of these problems is the failure of inverse limits to preserve exactness, as well as the failure of completion to be exact for non-finitely generated modules. The idea of using pro-groups to prove the completion theorem goes back to the original paper of Atiyah and Segal [AS69].

2.3 The case of a twisting arising from a graded central extension

In this section we will prove Theorem 2 in the case where the twisting τ arises from a central extension in the way explained in [FHT07b]. That is, we will prove the following.

Theorem 8. *Let X be a finite G-CW complex, where G is a compact Lie group. Then the projection $\pi : EG \times X \to X$ induces an isomorphism*

$$K^{\tau+*}_G(X)^{\widehat{}}_{I_G} \xrightarrow{\approx} K^{\pi^*(\tau)+*}_G(EG \times X)$$

for any twisting τ arising from a graded central extension

$$1 \to \mathbb{T} \to G^\tau \to G \to 1, \qquad \epsilon : G \to \mathbf{Z}/2.$$

Our argument for proving Theorem 8 is closely based on the one Adams, Haeberly, Jackowski and May present for proving a generalization of the Atiyah–Segal completion theorem in the untwisted case [AHJM88a]. Their argument in turn builds on ideas due to Carlsson [Car84]. As before, let

$$1 \to C \to \tilde{G} \to G \to 1$$

be a central extension of a compact Lie group G by a compact Lie group, and let π be an irreducible representation of C. We shall derive Theorem 8 from the following result.

Theorem 9. *Suppose X_1 and X_2 are G-spaces, and let $f : X_1 \to X_2$ be a G-equivariant map which is a non-equivariant homotopy equivalence. Then the map*

$$f^* : \mathbf{K}^*_{\tilde{G}}(X_2)(\pi)\widehat{_{I_G}} \to \mathbf{K}^*_{\tilde{G}}(X_1)(\pi)\widehat{_{I_G}}$$

is an isomorphism.

Before proving Theorem 9, we explain how it implies Theorem 8.

Proof of Theorem 8 assuming Theorem 9. Let Z be a finite G-CW complex. By Theorem 9, the projection map $\pi : EG \times Z \to Z$ induces an isomorphism

$$\mathbf{K}^*_{G^\tau}(Z)(1)\widehat{_{I_G}} \xrightarrow[\approx]{\pi^*} \mathbf{K}^*_{G^\tau}(EG \times Z)(1)\widehat{_{I_G}}. \tag{2.4}$$

Since Z is finite, we have

$$\begin{aligned} \mathbf{K}^*_{G^\tau}(Z)(1)\widehat{_{I_G}} &= \{K^*_{G^\tau}(Z_\alpha)(1)/I^n_G \cdot K^*_{G^\tau}(Z_\alpha)(1)\}_{\alpha,n} \\ &= \{K^*_{G^\tau}(Z)(1)/I^n_G \cdot K^*_{G^\tau}(Z)(1)\}_n. \end{aligned} \tag{2.5}$$

Fix a model for EG which is a countable ascending union of finite G-CW subcomplexes EG_k, $k \geq 1$; for example, we could take EG to be the iterated join construction of Milnor and take EG_k to be the k-fold join of G with itself. Then Lemma 6 and the finiteness of Z imply that

$$\begin{aligned} \mathbf{K}^*_{G^\tau}(EG \times Z)(1)^{\widehat{}}_{I_G} &= \mathbf{K}^*_{G^\tau}(EG \times Z)(1) \\ &= \{K^*_{G^\tau}(EG_k \times Z)(1)\}_k. \end{aligned} \tag{2.6}$$

Thus applying the limit functor taking pro-$R(G)$-modules to $R(G)$-modules to the isomorphism (2.4) gives us an isomorphism

$$K^*_{G^\tau}(Z)(1)^{\widehat{}}_{I_G} \xrightarrow[\approx]{\pi^*} \varprojlim_k K^*_{G^\tau}(EG_k \times Z)(1). \tag{2.7}$$

Using (2.6), (2.4) and (2.5), we see that inverse system $\{K^*_{G^\tau}(EG_k \times Z)(1)\}_k$ is equivalent to one that satisfies the Mittag–Leffler condition, whence the $\lim^1$ error terms vanish and the codomain in (2.7) is isomorphic to $K^*_{G^\tau}(EG \times Z)(1)$. Thus for any finite G-CW complex Z, we have a natural isomorphism

$$K^*_{G^\tau}(Z)(1)^{\widehat{}}_{I_G} \xrightarrow[\approx]{\pi^*} K^*_{G^\tau}(EG \times Z)(1).$$

Suppose now Z is a based finite G-CW complex. Then from the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{K}^*_{G^\tau}(Z)(1)^{\widehat{}}_{I_G} & \longrightarrow & K^*_{G^\tau}(Z)(1)^{\widehat{}}_{I_G} & \longrightarrow & K^*_{G^\tau}(\mathrm{pt})(1)^{\widehat{}}_{I_G} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \approx & & \downarrow \approx & & \\ 0 & \longrightarrow & \tilde{K}^*_{G^\tau}(EG_+ \wedge Z)(1) & \longrightarrow & K^*_{G^\tau}(EG \times Z)(1) & \longrightarrow & K^*_{G^\tau}(EG)(1) & \longrightarrow & 0 \end{array}$$

we see that there is an induced isomorphism

$$\tilde{K}^*_{G^\tau}(Z)(1)^{\widehat{}}_{I_G} \xrightarrow[\approx]{\pi^*} \tilde{K}^*_{G^\tau}(EG_+ \wedge Z)(1).$$

The claim now follows by taking Z to be the space $X_+ \wedge S^1(\epsilon)$ and applying Proposition 3. □

The rest of this section is dedicated to the proof of Theorem 9. Let $\{V_i\}_{i \in I}$ be a set of representatives for the isomorphism classes of the non-trivial irreducible complex representations of G. Then I is countable, the fixed-point subspace V_i^G is zero for each $i \in I$, and for every proper closed subgroup H of G, there is some $i \in I$ such that $V_i^H \neq 0$. Let U be the direct sum of countably many copies of $\bigoplus_{i \in I} V_i$, and let

$$Y = \operatorname{colim}_{V \subset U} S^V$$

where the colimit is over all finite-dimensional G-subspaces of U and S^V denotes the one-point compactification of V. Pick a G-invariant inner product on U, and observe that Y^G is S^0.

Lemma 10. *The space Y is H-equivariantly contractible for any proper closed subgroup H of G.*

Proof. Since Y has the structure of an H-CW complex, it is enough to show that the fixed point set Y^K is weakly equivalent to a point for any subgroup K of H. Given any finite-dimensional G-subspace $V \subset U$, we can find a finite-dimensional G-subspace $W \subset U$ such that $V \subset W$ and $(W - V)^K \neq 0$, where $W - V$ denotes the orthogonal complement of V in W. But then the inclusion $S^V \hookrightarrow S^W$ is K-equivariantly null-homotopic, whence the map $(S^V)^K \hookrightarrow (S^W)^K$ is null-homotopic. Since Y^K is given by the union

$$Y^K = \operatorname{colim}_{V \subset U} (S^V)^K,$$

the claim follows. □

Lemma 11. *The pro-$R(G)$-module $\tilde{\mathbf{K}}_{\tilde{G}}(Y)(\pi)^{\wedge}_{I_G}$ is pro-zero.*

Proof. For a finite-dimensional G-subspace $V \subset U$, let

$$\lambda_V \in \tilde{K}_G(S^V) = \tilde{K}_{\tilde{G}}(S^V)(0) \subset \tilde{K}_{\tilde{G}}(S^V)$$

denote the equivariant Bott class, where "0" refers to the trivial representation of C. Then by Bott periodicity, each element of $\tilde{K}_{\tilde{G}}(S^V)(\pi)$ is uniquely expressible as a product $x\lambda_V$, where $x \in \tilde{K}^*_{\tilde{G}}(S^0)(\pi)$. Suppose $W \supset V$. From the diagram

$$\begin{array}{ccc} \tilde{K}^*_{\tilde{G}}(S^{W-V})(\pi) & \longrightarrow & \tilde{K}_{\tilde{G}}(S^0)(\pi) \\ \approx \big\downarrow \wedge\lambda_V & & \approx \big\downarrow \wedge\lambda_V \\ \tilde{K}^*_{\tilde{G}}(S^W)(\pi) & \longrightarrow & \tilde{K}_{\tilde{G}}(S^V)(\pi) \end{array}$$

it follows that the map

$$\tilde{K}^*_{\tilde{G}}(S^W)(\pi) \to \tilde{K}_{\tilde{G}}(S^V)(\pi)$$

sends $x\lambda_W$ to $x\chi_{W-V}\lambda_V$, where χ_{W-V} denotes the image of λ_{W-V} under the map

$$\tilde{K}^*_G(S^{W-V}) \to \tilde{K}_G(S^0)$$

induced by the inclusion $S^0 \hookrightarrow S^{W-V}$. Since this map is non-equivariantly null-homotopic, it follows from the diagram

$$\begin{array}{ccccc} \tilde{K}_G(S^{W-V}) & \longrightarrow & \tilde{K}_G(S^0) & = & R(G) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \tilde{K}(S^{W-V}) & \longrightarrow & \tilde{K}(S^0) & = & \mathbf{Z} \end{array}$$

that $\chi_{W-V} \in I_G$. Thus if we choose $W \subset U$ so that it is the direct sum of V with n G-invariant subspaces of U, then the map

$$\tilde{K}^*_G(S^W)(\pi)/I^n_G \cdot \tilde{K}^*_G(S^W)(\pi) \to \tilde{K}^*_G(S^V)(\pi)/I^n_G \cdot \tilde{K}^*_G(S^V)(\pi)$$

is zero. It follows that for any fixed n the pro-$R(G)$-module

$$\{\tilde{K}_{\tilde{G}}(S^V)/I^n_G \cdot \tilde{K}_{\tilde{G}}(S^V)\}_V$$

is pro-zero, and therefore so is

$$\tilde{\mathbf{K}}_{\tilde{G}}(Y)(\pi)\widehat{\ }_{I_G} = \{\tilde{K}_{\tilde{G}}(S^V)/I^n_G \cdot \tilde{K}_{\tilde{G}}(S^V)\}_{n,V} = \varprojlim_n \{\tilde{K}_{\tilde{G}}(S^V)/I^n_G \cdot \tilde{K}_{\tilde{G}}(S^V)\}_V$$

where the inverse limit is taken in the category of pro-$R(G)$-modules. □

We are now ready to prove Theorem 9.

Proof of Theorem 9. It is enough to prove that $\tilde{\mathbf{K}}^*_{\tilde{G}}(Z)(\pi)\widehat{_{I_G}}$ is pro-zero when Z is a non-equivariantly contractible G-space; the claim then follows by taking Z to be the mapping cone of f. We shall show that $\tilde{\mathbf{K}}^*_{\tilde{G}}(Z)(\pi)\widehat{_{I_G}} = 0$ for such Z by induction on the subgroups of G, making use of the fact that any strictly descending chain of closed subgroups of a Lie group is of finite length.

To start the induction, we observe that in the case $G = \{e\}$ the claim follows from the assumption that Z is non-equivariantly contractible. Assume inductively that

$$\tilde{\mathbf{K}}^*_{\tilde{H}}(Z)(\pi)\widehat{_{I_H}} = 0$$

for all proper closed subgroups H of G; here as before $\tilde{H}$ denotes the inverse image of H in $\tilde{G}$. The inclusion of the fixed-point set $Y^G = S^0$ into Y gives a cofiber sequence

$$S^0 \to Y \to Y/S^0$$

whence we have a cofiber sequence

$$Z \to Z \wedge Y \to Z \wedge (Y/S^0).$$

Thus to show that $\tilde{\mathbf{K}}^*_{\tilde{G}}(Z)(\pi)\widehat{_{I_G}} = 0$, it is enough to show that

$$\tilde{\mathbf{K}}^*_{\tilde{G}}(Z \wedge Y)(\pi)\widehat{_{I_G}} = 0$$

and

$$\tilde{\mathbf{K}}^*_{\tilde{G}}(Z \wedge (Y/S^0))(\pi)\widehat{_{I_G}} = 0.$$

Let us first show that $\tilde{\mathbf{K}}^*_{\tilde{G}}(Z \wedge Y)(\pi)\widehat{_{I_G}} = 0$; we claim that in fact

$$\tilde{\mathbf{K}}^*_{\tilde{G}}(W \wedge Y)(\pi)\widehat{_{I_G}} = 0$$

for any based G-CW complex W. Observing that

$$\tilde{\mathbf{K}}^*_{\tilde{G}}(W \wedge Y)(\pi)\widehat{_{I_G}} = \varprojlim_\alpha \tilde{\mathbf{K}}^*_{\tilde{G}}(W_\alpha \wedge Y)(\pi)\widehat{_{I_G}}$$

where W_α runs through all finite G-CW complexes of W, we see that it is enough to consider the case where W is finite. Filtering W by skeleta and working inductively reduces us to the case where W is of the form $G/H_+ \wedge S^n$ for some n and some closed subgroup H of G, and using the suspension axiom further reduces us to the case $W = G/H_+$. But now in the case $H = G$ the claim follows from Lemma 11; and in the case $H \lneq G$, it follows from the change of groups isomorphism (Lemma 5)

$$\tilde{\mathbf{K}}^*_{\tilde{G}}(G/H_+ \wedge Y)(\pi)\widehat{_{I_G}} \approx \tilde{\mathbf{K}}^*_{\tilde{H}}(Y)(\pi)\widehat{_{I_H}}$$

together with Lemma 10.

It remains to show that $\tilde{\mathbf{K}}^*_{\tilde{G}}(Z \wedge (Y/S^0))(\pi)\widehat{_{I_G}} = 0$. We shall show that in fact

$$\tilde{\mathbf{K}}^*_{\tilde{G}}(Z \wedge W)(\pi)\widehat{_{I_G}} = 0$$

for any based G-CW complex W such that W^G is a point. Arguing as above, we see that it is enough to consider W of the form $W = G/H_+$, where H now has to be a proper closed subgroup of G. But in this case the claim follows from the change of groups isomorphism (Lemma 5)

$$\tilde{\mathbf{K}}^*_{\tilde{G}}(Z \wedge G/H_+)(\pi)\widehat{_{I_G}} = \tilde{\mathbf{K}}^*_{\tilde{H}}(Z)(\pi)\widehat{_{I_H}}$$

and the inductive assumption. □

2.4 The general case

In this section we finally prove Theorem 2 in full generality. We shall do so by considering successively more general spaces, starting with the case $X = \mathrm{pt}$ and proceeding by change of groups and Mayer–Vietoris arguments. Since in general completion is exact only for finitely generated modules, along the way we check that the twisted K-groups that enter the Mayer–Vietoris sequences are finitely generated over $R(G)$.

Lemma 12. *Theorem 2 holds and $K_G^{\tau+*}(X)$ is finitely generated over $R(G)$ when $X = \mathrm{pt}$.*

Proof. By [FHT07b], Example 2.29, any twisting of a point arises from a graded central extension. Thus Theorem 8 shows that Theorem 2 holds in this case. The claim about finite generation follows from Proposition 3 and the proof of Proposition 4. □

Lemma 13. *Theorem 2 holds and $K_G^{\tau+*}(X)$ is finitely generated over $R(G)$ when $X = G/H$, where H is a closed subgroup of G.*

Proof. Notice that $G/H = G \times_H \mathrm{pt}$ and that $EG \times G/H = G \times_H EG$. For any H-space Z, we have a natural local equivalence of topological groupoids

$$Z//H \to G \times_H Z//G$$

giving rise to a natural change of groups isomorphism

$$K_G^{\tau+*}(G \times_H Z) \xrightarrow{\approx} K_H^{\tau+*}(Z). \tag{2.8}$$

Consider the diagram

$$\begin{array}{ccc} K_G^{\tau+*}(G/H)\widehat{}_{I_G} & \longrightarrow & K_H^{\tau+*}(\mathrm{pt})\widehat{}_{I_H} \\ \downarrow & & \downarrow \\ K_G^{\pi^*(\tau)+*}(EG \times G/H) & \longrightarrow & K_H^{\pi^*(\tau)+*}(EG) \end{array}$$

Here the bottom row is a change of groups isomorphism as in (2.8); the top row is an isomorphism because of the isomorphism (2.8) and the fact that I_H-adic and I_G-adic completions of an $R(H)$-module agree (see [Seg68], Corollary 3.9); and the vertical map on the right is an isomorphism by Lemma 12 and the observation that EG is a model for EH. Thus the map on the left is also an isomorphism, which shows that Theorem 2 holds in this case. To see that $K_G^{\tau+*}(G/H)$ is finitely generated as an $R(G)$-module, observe that the isomorphism (2.8) and Lemma 12 imply that it is finitely generated over $R(H)$. The claim now follows from the fact that $R(H)$ is finite over $R(G)$ (see [Seg68], Proposition 3.2). □

Lemma 14. *Theorem 2 holds and* $K_G^{\tau+*}(X)$ *is finitely generated over* $R(G)$ *when* X *is of the form* $X = G/H \times S^n$, $n \geq 0$.

Proof. The case where $n = 0$ follows from Lemma 13 and the axiom of disjoint unions. For $n > 0$, the claim follows inductively from the Mayer–Vietoris sequences arising from the decomposition of S^n into upper and lower hemispheres S^n_+ and S^n_-. Lemma 13 and the inductive assumption imply that all groups in the Mayer–Vietoris sequence

$$\begin{aligned} \cdots \to K_G^{\tau+*}(G/H \times S^n) \to \\ \to K_G^{\tau+*}(G/H \times S^n_+) \oplus K_G^{\tau+*}(G/H \times S^n_+) \to \\ \to K_G^{\tau+*}(G/H \times (S^n_+ \cap S^n_-)) \to \cdots \end{aligned} \tag{2.9}$$

except $K_G^{\tau+*}(G/H \times S^n)$ are finitely generated over $R(G)$, whence $K_G^{\tau+*}(G/H \times S^n)$ must also be finitely generated, as claimed. It follows that the sequence obtained from (2.9) by completion with respect to the augmentation ideal I_G is exact. Now the claim that Theorem 2 holds for the space $G/H \times S^n$ follows from Lemma 13 and the inductive assumption by comparing the completed sequence to the Mayer–Vietoris sequence of the pair

$$(EG \times G/H \times S^n_+, EG \times G/H \times S^n_-)$$

and applying the five lemma. □

Theorem 2 is now contained in the following.

Theorem 15. *Theorem 2 holds and* $K_G^{\tau+*}(X)$ *is finitely generated over* $R(G)$ *for any finite G-CW complex* X.

Proof. We proceed by induction on the number of cells in X. If X has no cells, that is, if X is the empty G-space, the claim holds trivially. Assume inductively that the claim holds for the space X, and consider the space $Y = X \cup_f (G/H \times D^n)$, where

$f : G/H \times S^{n-1} \to X$ is an attaching map. Denote

$$D^n(r) = \{x \in \mathbf{R}^n : |x| \leq r\},$$

and let

$$Y_1 = X \cup_f (G/H \times (D^n - D^n(1/3))) \subset Y$$

and

$$Y_2 = D^n(2/3) \subset Y.$$

By Lemma 13, Lemma 14 and the inductive assumption, in the Mayer–Vietoris sequence

$$\cdots \longrightarrow K_G^{\tau+*}(Y) \longrightarrow K_G^{\tau+*}(Y_1) \oplus K_G^{\tau+*}(Y_2) \longrightarrow K_G^{\tau+*}(Y_1 \cap Y_2) \longrightarrow \cdots \quad (2.10)$$

all groups except possibly $K_G^{\tau+*}(Y)$ are finitely generated over $R(G)$. It follows that $K_G^{\tau+*}(Y)$ is also finitely generated, as claimed. We conclude that the top row in the following diagram of Mayer–Vietoris sequences is exact.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & K_G^{\tau+*}(Y)\hat{}_{I_G} & \longrightarrow & K_G^{\tau+*}(Y_1)\hat{}_{I_G} \oplus K_G^{\tau+*}(Y_2)\hat{}_{I_G} & \longrightarrow & K_G^{\tau+*}(Y_1 \cap Y_2)\hat{}_{I_G} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow \approx & & \downarrow \approx & & \\
\cdots & \longrightarrow & K_G^{\tau+*}(EG \times Y) & \longrightarrow & K_G^{\tau+*}(EG \times Y_1) \oplus K_G^{\tau+*}(EG \times Y_2) & \longrightarrow & K_G^{\tau+*}(EG \times (Y_1 \cap Y_2)) & \longrightarrow & \cdots
\end{array}$$

In the diagram the vertical map on the right is an isomorphism by Lemma 14 and the map in the middle is an isomorphism by Lemma 13 and the inductive assumption. Thus the map on the left is an isomorphism by the five lemma, showing that Theorem 2 holds for the space Y. □

We conclude the chapter with a brief note on an application of Theorem 2. The theorem implies that for a finite G-CW complex X and a twisting τ corresponding to an element of

$$H_G^1(X; \mathbf{Z}/2) \oplus H_G^3(X; \mathbf{Z}) = H^1(EG \times_G X; \mathbf{Z}/2) \oplus H_G^3(EG \times_G X; \mathbf{Z}),$$

there is an isomorphism

$$K_G^{\tau+*}(X)^{\widehat{}}_{I_G} \xrightarrow{\approx} K^{\tau+*}(EG \times_G X).$$

Denoting by G^{ad} the group G equipped with the conjugation action, and remembering that the space $EG \times_G G^{\mathrm{ad}}$ is homotopy equivalent to the free loop space LBG, we in particular obtain an isomorphism

$$K_G^{\tau+*}(G^{\mathrm{ad}})^{\widehat{}}_{I_G} \xrightarrow{\approx} K^{\tau+*}(LBG),$$

suggesting that Theorem 2 could be used to connect the twisted equivariant K-groups $K_G^{\tau+*}(G^{\mathrm{ad}})$ to the string topology of BG. Indeed this is the case: Kriz, Westerland and Levin have successfully used Theorem 2 to connect the groups $K_G^{\tau+*}(G^{\mathrm{ad}})$ with the Gruher–Salvatore [GS08] string topology of BG. See [KWL09], Theorem 28.

Chapter 3

Background on parametrized homotopy theory

In this chapter we will discuss background material on parametrized homotopy theory needed in the sequel. Our aim is to be relatively brief and simply explain the structures present in the parametrized context in enough detail to enable the reader to follow the subsequent sections, while also fixing notation that will be used later. In parametrized homotopy theory, for a given base space B, at least four different categories of interest arise: the category of (based) spaces parametrized by B; the category of spectra parametrized by B; and the respective homotopy categories of the two. We will discuss the point-set level category of parametrized based spaces in greatest detail, as this case is the most basic one and provides a good foundation for intuition. As we will explain, much of the structure present in the other categories is very similar.

For the most part of the discussion, we will follow May and Sigurdsson's monograph [MS06]. However, in the last section of the chapter we will also mention an alternative ∞-categorical approach to stable parametrized homotopy theory discussed by Ando, Blumberg, Gepner, Hopkins and Rezk in [ABG$^+$08] and [ABG10].

3.1 Parametrized spaces and ex-spaces

The analogue of a space in parametrized homotopy theory is a space X over a space B, that is, a continuous map of spaces

$$X \xrightarrow{p} B.$$

We call X the *total space* and B the *base space*, and think of X as a family of spaces, namely the fibers of p, parametrized by the points of B and glued together by the topology of X. A map between parametrized spaces $X \xrightarrow{p} B$ and $Y \xrightarrow{q} B$ is simply a map $f : X \to Y$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & B & \end{array}$$

commutative. Similarly, the parametrized analogue of a space equipped with a basepoint is a space $X \xrightarrow{p} B$ over B equipped with a section $s : B \to X$ of the map p. We call such an object an *ex-space* over B, and think of it as a family of based spaces, the section giving a basepoint for each fiber of p. A map between ex-spaces over B is a map of the underlying spaces over B respecting the sections. For technical reasons, when considering spaces or ex-spaces over B, we will restrict the total space to be a k-space and the base space to be a compactly generated space. We will denote the categories of such spaces over B and ex-spaces over B by $\mathscr{K}/B$ and $\mathscr{K}_B$, respectively. In what follows, we will mostly focus on the category of ex-spaces. Analogously to the unparametrized case where we can easily turn an unbased space into a based one by adding a disjoint basepoint, given a space $X \xrightarrow{p} B$ over B, we can form an ex-space over B by adding a disjoint section. The total space of this construction is $X \amalg B$, and the projection and the section maps are the obvious ones. We will denote this ex-space by $(X, p)_+$

The category $\mathscr{K}_B$ of ex-spaces over B admits smash products $X \wedge_B Y$ and internal hom objects $F_B(X, Y)$ making it into a closed symmetric monoidal category. Denoting by Z_b the fiber of an ex-space $Z \in \mathscr{K}_B$ over a point $b \in B$, these constructions satisfy

$$(X \wedge_B Y)_b = X_b \wedge Y_b \quad \text{and} \quad F_B(X,Y)_b = F(X_b, Y_b).$$

The identity element for the smash product is the parametrized 0-sphere S^0_B with total space $B \amalg B$, projection the folding map $B \amalg B \to B$ and section the inclusion of one copy of B.

In addition to the closed monoidal structure, the category $\mathscr{K}_B$ has the structure of a category enriched, tensored and cotensored over based spaces. If $X, Y \in \mathscr{K}_B$, we topologize the set of maps $\mathscr{K}_B(X,Y)$ as a subspace of the space of all maps from the total space of X to the total space of Y. The basepoint is given by the composite of the projection $X \to B$ and the section $B \to Y$. The tensors and cotensors can be conveniently described in terms of the closed monoidal structure. Given a based space K, equipping the product $B \times K$ with the obvious projection and section makes it into an ex-space over B, and then the smash product of an ex-space X and the space K and the ex-space of maps from K to X are given by

$$X \wedge_B K = X \wedge_B (B \times K) \quad \text{and} \quad F_B(K,X) = F_B(B \times K, X)$$

respectively.

An important feature of parametrized homotopy theory is the existence of various base-change functors; it is from these functors and their relations to each other that parametrized homotopy theory derives much of its force. Perhaps the most basic of the base-change functors is the pullback functor: given a map $f : A \to B$ of base spaces and an ex-space $B \xrightarrow{s} X \xrightarrow{p} B$ over B, we can define an ex-space f^*X over A by the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ f^*X & \longrightarrow & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{f} & B \end{array}$$

where the bottom square is a pullback square and the section t is defined by the pullback property and the requirement that qt is the identity map of A. The pullback functor

$$f^* : \mathscr{K}_B \to \mathscr{K}_A$$

thus defined admits both a left adjoint $f_!$ and a right adjoint f_*. If $A \to X \to A$ is an ex-space over A, then the ex-space $f_!X$ over B is defined by the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ X & \longrightarrow & f_!X \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

where the top square is a pushout square and the projection map $f_!X \to B$ is defined by the pushout property and the requirement that the composite of the right-hand vertical maps should be the identity map of B. The precise definition of the ex-space f_*X is slightly more involved, and we will simply point out that the fiber of f_*X over $b \in B$ is the space of sections of the map $X_b \to A_b$, where A_b and X_b denote the fibers over b of the maps $A \xrightarrow{f} B$ and $X \to A \xrightarrow{f} B$, respectively.

Example 16. It is instructive to consider the behavior of the base-change functors associated to the constant map r from a space B to the one-point space. An ex-space over a point is simply a based space K, and the pullback r^*K is simply the ex-space $B \times K$ over B we encountered previously when describing the tensor and cotensor structure of $\mathscr{K}_B$. If X is an ex-space over B with section s, then $r_!X$ is the space X/B obtained from X by collapsing the subspace sB of X into a point, while r_*X is the space of sections for the projection map $X \to B$, with the section s serving as the basepoint.

Example 17. If $\xi \to B$ is a vector bundle, then forming the fiberwise one-point compactification of ξ, we obtain a sphere bundle $S^\xi \to B$, which we make into an ex-space by equipping it with the section $B \to S^\xi$ given by the points at infinity. Now

$r_!S^\xi$ is precisely the Thom space B^ξ of ξ. Moreover, the space $r_!r^*r_!S^\xi$ is naturally homeomorphic to $B^\xi \wedge B_+$ (see (3.7) below), and the unit map

$$B^\xi \approx r_!S^\xi \xrightarrow{r_!(\eta)} r_!r^*r_!S^\xi \approx B^\xi \wedge B_+$$

corresponds to the Thom diagonal.

It can be shown that the for $f : A \to B$, the pullback functor $f^* : \mathscr{K}_B \to \mathscr{K}_A$ is closed symmetric monoidal in the sense that there exist coherent natural isomorphisms

$$f^*S^0_B \approx S^0_A, \quad f^*(X \wedge_B Y) \approx f^*X \wedge_A f^*Y \quad \text{and}$$
$$f^*F_B(X,Y) \approx F_A(f^*X, f^*Y).$$

The natural isomorphisms in the following proposition then follow formally from the results in [FHM03].

Proposition 18 (See [MS06], Proposition 2.2.2)**.** *Let $f : A \to B$ be a map, and let X be an ex-space over A and Y and Z be ex-spaces over B. Then we have the following natural isomorphisms:*

$$f^*S^0_B \approx S^0_A \tag{3.1}$$
$$f^*(Y \wedge_B Z) \approx f^*Y \wedge_A f^*Z \tag{3.2}$$
$$F_B(Y, f_*X) \approx f_*F_A(f^*Y, X) \tag{3.3}$$
$$f^*F_B(Y,Z) \approx F_A(f^*Y, f^*Z) \tag{3.4}$$
$$f_!(f^*Y \wedge_A X) \approx Y \wedge_B f_!X \tag{3.5}$$
$$F_B(f_!X, Y) \approx f_*F_A(X, f^*Y) \tag{3.6}$$

In the above proposition, the isomorphisms (3.1), (3.2) and (3.4) are part of the statement that f^* is closed symmetric monoidal, and (3.3) and (3.6) are adjunction relations. The remaining isomorphism (3.5) is sometimes called the *projection formula*. Observing that $f_!S^0_A \approx (A,f)_+$, as a special cases of (3.5) and (3.6) we have the isomorphisms

$$f_! f^* Y \approx Y \wedge_B (A, f)_+ \quad \text{and} \quad f_* f^* Y \approx F_B((A, f)_+, Y). \tag{3.7}$$

The base-change functors also satisfy various commutation relations. The following proposition is essentially [MS06], Proposition 2.2.11.

Proposition 19. *Suppose we have a pullback square of spaces*

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ A & \xrightarrow{f} & B \end{array}$$

Then the mates of the natural isomorphism $i^* f^* \approx g^* j^*$ with respect to the adjunctions $(f_!, f^*)$ and $(g_!, g^*)$ on one hand and with respect to the adjunctions (j^*, j_*) and (i^*, i_*) on the other give natural isomorphisms

$$g_! i^* \approx j^* f_! \quad \textit{and} \quad f^* j_* \approx i_* g^*,$$

respectively.

More explicitly, the first isomorphism in the preceding proposition is given by the composite

$$g_! i^* \to g_! i^* f^* f_! \approx g_! g^* j^* f_! \to j^* f_!$$

where the first map is given by the unit $\mathrm{id} \to f^* f_!$, the second one by the isomorphism $i^* f^* \approx g^* j^*$ and the third by the counit $g_! g^* \to \mathrm{id}$; while the second isomorphism in the proposition is given by the composite

$$f^* j_* \to i_* i^* f^* j_* \approx i_* g^* j^* j_* \to i_* g^*$$

where the first map is given by the unit $\mathrm{id} \to i_* i^*$ and the last one by the counit $j^* j_* \to \mathrm{id}$. Of course, by symmetry we now have similar isomorphisms $i_! g^* \approx f^* j_!$ and $j^* f_* \approx g_* i^*$ as well.

We now turn attention to the structure present on the level of the homotopy categories of ex-spaces, and assume that our base spaces are homotopy equivalent to CW complexes in addition to being compactly generated. Each category $\mathscr{K}_B$ admits a Quillen model structure in which weak equivalences are the maps that are weak equivalences between the total spaces; for an ex-space to be fibrant with respect to these model structures it is sufficient that the projection of the ex-space is a Serre fibration. With respect to these model structures, the base-change adjunctions $(f_!, f^*)$ are Quillen adjunctions, and they are Quillen equivalences when f is a homotopy equivalence. Thus the adjoint pair $(f_!, f^*)$ induces an adjoint pair of derived functors between the homotopy categories, and we will continue to write $f_!$ and f^* for the derived versions. There is also a derived version of the smash product $\wedge_B$ which makes the homotopy category $\operatorname{Ho}\mathscr{K}_B$ a symmetric monoidal category. We will continue to denote this derived smash product by $\wedge_B$. The derived pullback functors f^* are then symmetric monoidal functors.

In addition to the preceding structure, one can also construct derived versions of the base-change functor f_* and the mapping ex-spaces $F_B(X, Y)$, but their construction is more complicated. The constructions May and Sigurdsson present rely on a version of the Brown representability theorem, and due to restrictions inherent in this method, the derived f_*Y and $F_B(X, Y)$ are defined only when the ex-space Y is connected in the sense that all fibers of a fibrant replacement of Y are connected. Restricting to connected ex-spaces where necessary, the analogues of Propositions 18 and 19 then continue to hold on the level of homotopy categories of ex-spaces if in the case of Proposition 19 we make the additional assumption that either f or j is a Serre fibration. See [MS06], Section 9.4.

One can show that a map $f : X \to Y$ between fibrant ex-spaces is a weak equivalence precisely when it restricts to a weak equivalence between all fibers. Thus we have the following criterion for a map of ex-spaces to be a weak equivalence, where for a point $b \in B$ we use b to denote the map from a one-point space into B whose image is the point b.

Proposition 20. *A map $f : X \to Y$ in $\operatorname{Ho}\mathscr{K}_B$ is an equivalence if and only if for* *every $b \in B$, the derived pullback of f*

$$b^*(f) : b^*X \to b^*Y$$

is an equivalence in $\operatorname{Ho} \mathscr{K}_*$.

3.2 Parametrized spectra

We now turn our attention to stable parametrized homotopy theory. We continue to assume that our base spaces are homotopy equivalent to CW complexes. In simplest terms, a parametrized (pre)spectrum X over a space B consists of a sequence of ex-spaces $X_0, X_1, \ldots$ over B together with maps of ex-spaces

$$\Sigma_B X_i \to X_{i+1}$$

where Σ_B denotes fiberwise suspension. However, a great deal more care is required to obtain a category of parametrized spectra with a well-behaved fiberwise smash product $\wedge_B$, and May and Sigurdsson achieve this by generalizing orthogonal spectra into the parametrized setting. We will denote the category of spectra over B by $\mathscr{S}_B$. As already indicated, this category admits smash products $X \wedge_B Y$, and furthermore there are parametrized function spectra $F_B(X, Y)$ which together with the smash product make $\mathscr{S}_B$ into a closed symmetric monoidal category. The unit for the smash product is the parametrized sphere spectrum S_B. Again, a map $f : A \to B$ induces a pull-back functor

$$f^* : \mathscr{S}_B \to \mathscr{S}_A,$$

which is closed symmetric monoidal and has a left adjoint $f_!$ and a right adjoint f_*. Moreover, the analogues of Propositions 18 and 19 hold for parametrized spectra (see [MS06], Section 11.4).

The category of ex-spaces and spectra over B are related to each other by an adjoint pair of functors

$$\Sigma_B^\infty : \mathscr{K}_B \rightleftarrows \mathscr{S}_B : \Omega_B^\infty$$

where Σ_B^∞ is a symmetric monoidal functor in the sense that there exist natural isomorphisms

$$\Sigma_B^\infty S_B^0 \approx S_B \quad \text{and} \quad \Sigma_B^\infty(K \wedge_B L) \approx \Sigma_B^\infty K \wedge_B \Sigma_B^\infty L$$

for $K, L \in \mathscr{K}_B$. Furthermore, Σ_B^∞ commutes with the base-change functors f^* and $f_!$ so that for a map $f : A \to B$ and ex-spaces $K \in \mathscr{K}_B$ and $L \in \mathscr{K}_A$, we have natural isomorphisms

$$\Sigma_A^\infty f^* K \approx f^* \Sigma_B^\infty K \quad \text{and} \quad \Sigma_B^\infty f_! L \approx f_! \Sigma_A^\infty L.$$

The categories $\mathscr{S}_B$ admit stable model structures making the base-change adjunctions $(f_!, f^*)$ into Quillen adjunctions that are Quillen equivalences when f is a homotopy equivalence. As in the case of ex-spaces, we continue to write $f_!$ and f^* for the derived versions. Again, there is a derived version of $\wedge_B$, and May and Sigurdsson use a version of the Brown representability theorem to construct a derived base-change functor f_* that is right adjoint to f^* as well as derived parametrized function spectra $F_B(X, Y)$. No connectedness assumptions like the ones that were needed in the case of ex-spaces are necessary in the stable situation. Together with the smash product, these function spectra make $\operatorname{Ho}\mathscr{S}_B$ into a closed symmetric monoidal category, and the derived pull-back functors f^* are closed symmetric monoidal. Again, the analogues of Propositions 18 and 19 hold as long as in the case of Proposition 19 we assume that at least one of f or j is a Serre fibration; see [MS06], Section 13.7. Also, the analogue of Proposition 20 holds: a map of parametrized spectra is a stable equivalence precisely when it restricts to a stable equivalence between all derived fibers.

The adjoint pair $(\Sigma_B^\infty, \Omega_B^\infty)$ is a Quillen adjunction, and hence descends to an adjunction

$$\Sigma_B^\infty : \operatorname{Ho}\mathscr{K}_B \rightleftarrows \operatorname{Ho}\mathscr{S}_B : \Omega_B^\infty$$

between the homotopy categories. For ex-spaces $K, L \in \operatorname{Ho}\mathscr{K}_B$, there are natural equivalences

$$\Sigma_B^\infty S_B^0 \simeq S_B \quad \text{and} \quad \Sigma_B^\infty(K \wedge_B L) \simeq \Sigma_B^\infty K \wedge_B \Sigma_B^\infty L.$$

Furthermore, the derived Σ_B^∞ commutes with the derived base-change functors f^* and $f_!$ so that for a map $f : A \to B$ and ex-spaces $K \in \mathscr{K}_B$ and $L \in \mathscr{K}_A$, there are natural equivalences

$$\Sigma_A^\infty f^* K \simeq f^* \Sigma_B^\infty K \quad \text{and} \quad \Sigma_B^\infty f_! L \simeq f_! \Sigma_A^\infty L.$$

3.3 The bicategories $\mathscr{E}x$ and $\mathscr{E}x_B$

It is possible to organize the various categories $\mathrm{Ho}\,\mathscr{S}_B$ for varying base spaces B into a closed bicategory $\mathscr{E}x$, and doing so will facilitate the discussion of the duality underlying the construction of the umkehr maps we will need. Recall that a *bicategory* $\mathscr{C}$ consists of the following data: first, a collection of objects; second, for each pair (A, B) of objects, a category $\mathscr{C}(A, B)$; third, for each object A, an assignment of a *unit object* $U_A \in \mathscr{C}(A, A)$; and fourth, composition functors

$$\odot : \mathscr{C}(B, C) \times \mathscr{C}(A, B) \to \mathscr{C}(A, C)$$

that are associative and left and right unital up to specified coherent natural isomorphisms, with the unit objects U_A serving as the units. We call the objects of $\mathscr{C}$ the 0-cells of $\mathscr{C}$, the objects of the categories $\mathscr{C}(A, B)$ the 1-cells $\mathscr{C}$, and the morphisms in these categories the 2-cells of $\mathscr{C}$. A monoidal category is precisely a bicategory with a single 0-cell, with the composition $\odot$ corresponding to the tensor product, so we can regard bicategories as multi-object generalizations of monoidal categories. The prototypical example of a bicategory of the type we are interested in is the bicategory $\mathscr{M}$ of bimodules, where the 0-cells are given by rings, and for rings R and S, the category of $\mathscr{M}(R, S)$ of 1-cells from R to S is the category of S-R-bimodules. The composition $\odot$ is given by tensor product, so that the composite of 1-cells ${}_SM_R : R \to S$ and ${}_TN_S : S \to T$ is the T-R-bimodule

$${}_TN_S \otimes_S {}_SM_R.$$

Roughly speaking, an *involution* t on a bicategory $\mathscr{C}$ consists of, first, a bijection

t on the collection of 0-cells such that $t^2 = \mathrm{id}$; second, equivalences of categories

$$t : \mathscr{C}(A, B) \to \mathscr{C}(B, A)$$

together with natural isomorphisms $tt \approx \mathrm{id}$; and third, coherent natural isomorphisms

$$tU_A \approx U_{tA} \quad \text{and} \quad t(Y \odot X) \approx tX \odot tY$$

for all 0-cells A and composeable 1-cells X and Y. We refer to [MS06], Definition 16.2.1 for a more precise definition. A *symmetric* bicategory is a bicategory equipped with an involution. For example, the bicategory $\mathscr{M}$ admits an involution $(\cdot)^{\mathrm{op}}$, and hence can be made into a symmetric bicategory. On the level of 0-cells, $(\cdot)^{\mathrm{op}}$ sends a ring R to the opposite ring R^{op}, while on the level of 1-cells $(\cdot)^{\mathrm{op}}$ sends an S-R-bimodule M to the same module M regarded as an R^{op}-S^{op}-bimodule.

A symmetric bicategory $\mathscr{C}$ is *closed* if for every triple (A, B, C) of 0-cells, there is a functor

$$\rhd : \mathscr{C}(A, B)^{\mathrm{op}} \times \mathscr{C}(A, C) \to \mathscr{C}(B, C)$$

such that for all 1-cells $X : A \to B$, $Y : B \to C$ and $Z : A \to C$, there is a natural isomorphism

$$\mathscr{C}(Y \odot X, Z) \approx \mathscr{C}(Y, X \rhd Z).$$

(Here we have used the notation $\mathscr{C}(Y \odot X, Z)$ to denote the set of 2-cells from $Y \odot X$ to Z, and similarly for $\mathscr{C}(Y, X \rhd Z)$. In general, we will rely on the context to make clear whether an expression of the form $\mathscr{C}(-, -)$ should be interpreted as a category of 1 and 2-cells between two 0-cells or as a set of 2-cells between two 1-cells.) Setting $Z \lhd Y = t(tY \rhd tZ)$, we then obtain functors

$$\lhd : \mathscr{C}(A, C) \times \mathscr{C}(B, C)^{\mathrm{op}} \to \mathscr{C}(A, B)$$

such that there are natural isomorphisms

$$\mathscr{C}(Y \odot X, Z) \approx \mathscr{C}(X, Z \lhd Y).$$

when X, Y and Z are as before. We think of $X \rhd Z$ and $Z \lhd Y$ as "hom"-1-cells, and call the adjoints

$$\varepsilon : (X \rhd Z) \odot X \to Z \quad \text{and} \quad \varepsilon : Y \odot (Z \lhd Y) \to Z \tag{3.8}$$

of the identity maps of $X \rhd Z$ and $Z \lhd Y$, respectively, *evaluation maps*. For example, the bicategory $\mathscr{M}$ is closed, and $\rhd$ and $\lhd$ in this bicategory are given by actual hom-functors: if ${}_S M_R : R \to S$, ${}_T N_S : S \to T$ and ${}_T L_R : R \to T$ are bimodules, then bimodule maps

$${}_T N_S \otimes_S {}_S M_R \to {}_T L_R$$

are in bijective correspondence with bimodule maps

$${}_T N_S \to \mathrm{Hom}_{\text{right}-R}({}_S M_R, {}_T L_R)$$

so that we can take

$$M \rhd L = \mathrm{Hom}_{\text{right}-R}({}_S M_R, {}_T L_R),$$

and then

$$L \lhd N = \mathrm{Hom}_{\text{left}-T}({}_T N_S, {}_T L_R).$$

The evaluation maps (3.8) in this case correspond to evaluation of homomorphisms against arguments, justifying our terminology.

Having introduced the relevant definitions, we will now describe the closed bicategory $\mathscr{E}x$. The objects of $\mathscr{E}x$ are spaces which are compactly generated and have the homotopy type of a CW complexes, and given spaces A and B, the category of $\mathscr{E}x(A, B)$ of 1-cells from A to B is $\mathrm{Ho}\,\mathscr{S}_{B\times A}$. Given 1-cells $X : A \to B$, $Y : B \to C$ and $Z : A \to C$, we have

$$Y \odot X \simeq (\pi^{CBA}_{CA})_! \left((\pi^{CBA}_{CB})^* Y \wedge_{C\times B\times A} (\pi^{CBA}_{BA})^* X\right), \tag{3.9}$$

$$X \rhd Z \simeq (\pi^{CBA}_{CB})_* F_{C\times B\times A} \left((\pi^{CBA}_{BA})^* X, (\pi^{CBA}_{CA})^* Z\right) \tag{3.10}$$

and

$$Z \lhd Y \simeq (\pi^{CBA}_{BA})_* F_{C\times B\times A} \left((\pi^{CBA}_{CB})^* Y, (\pi^{CBA}_{CA})^* Z\right) \tag{3.11}$$

where the various maps called π stand for projection maps with the indicated domains and codomains (for brevity, we have omitted the $\times$-signs). The unit 1-cell $U_A \in \mathscr{E}x(A, A)$ is given by $\Delta_! S_A$, where $\Delta : A \to A \times A$ is the diagonal map. The symmetry involution t is the identity on 0-cells, and on 1-cells we have

$$t = \tau^* : \mathrm{Ho}\,\mathscr{S}_{B\times A} \to \mathrm{Ho}\,\mathscr{S}_{A\times B}$$

where $\tau : A \times B \to B \times A$ is the map that interchanges the A and B coordinates.

The closed bicategories $\mathscr{E}x_B$ are analogues of $\mathscr{E}x$ that are suitable for work relative to a base space B. The objects of $\mathscr{E}x_B$ are fibrations $K \to B$, $L \to B$, and so on with base space B, and the category of 1-cells from $K \to B$ to $L \to B$ is $\mathrm{Ho}\,(\mathscr{S}_{L\times_B K})$. For 1-cells $X : K \to L$, $Y : L \to M$ and $Z : K \to M$ (we now omit the projection to B from the notation for a 0-cell), we have

$$Y \odot_B X \simeq (\pi^{MLK}_{MK})_! \left((\pi^{MLK}_{ML})^* Y \wedge_{M\times_B L\times_B K} (\pi^{MLK}_{LK})^* X\right), \tag{3.12}$$

$$X \rhd_B Z \simeq (\pi^{MLK}_{ML})_* F_{M\times_B L\times_B K} \left((\pi^{MLK}_{LK})^* X, (\pi^{MLK}_{MK})^* Z\right) \tag{3.13}$$

and

$$Z \lhd_B Y \simeq (\pi^{MLK}_{LK})_* F_{M\times_B L\times_B K} \left((\pi^{MLK}_{ML})^* Y, (\pi^{MLK}_{MK})^* Z\right). \tag{3.14}$$

The unit $U_K \in \mathscr{E}x_B(K, K)$ is given by $\delta_! S_K$, where $\delta : K \to K \times_B K$ is the diagonal map. The symmetry involution t is again the identity on 0-cells and is induced by the coordinate exchange maps $K \times_B L \to L \times_B K$ for 1-cells and 2-cells. The bicategory $\mathscr{E}x$ is the special case $\mathscr{E}x_{\mathrm{pt}}$. We warn the reader that in their definition of $\mathscr{E}x_B$, May and Sigurdsson do not require the maps $K \to B$ to be fibrations for objects of $\mathscr{E}x_B$, and our $\mathscr{E}x_B$ then corresponds to the full sub-bicategory $\mathscr{E}x^{fib}_B$ of their $\mathscr{E}x_B$. Since it is this sub-bicategory $\mathscr{E}x^{fib}_B$ that will be relevant to our work, we have opted to give it the simpler name $\mathscr{E}x_B$

Given a spectrum X over a space B, we can interpret X as either a 1-cell

$$X : B \to \mathrm{pt}$$

or a 1-cell

$$X : \mathrm{pt} \to B$$

in $\mathscr{E}x$, and we choose the first interpretation as our default. The 1-cell tX then corresponds to X interpreted in the second way. If Y is another spectrum over B, then (3.9), (3.10) and (3.11) give

$$Y \odot tX \simeq r_!(Y \wedge_B X) \quad \text{and} \quad X \rhd Y \simeq tY \lhd tX \simeq r_* F_B(X, Y)$$

where $r : B \to \mathrm{pt}$ is the unique map from B to the one-point space. Similarly, given a fibration $K \xrightarrow{p} B$ and a parametrized spectrum X over K, we will interpret X as a 1-cell

$$X : K \to B$$

in $\mathscr{E}x_B$, where in the codomain we have written B for the identity fibration $\mathrm{id} : B \to B$. The 1-cell tX is then X interpreted as a 1-cell $B \to K$, and if Y is another spectrum over K, then (3.12), (3.13) and (3.14) give the equivalences

$$Y \odot_B tX \simeq p_!(Y \wedge_K X) \quad \text{and} \quad X \rhd_B Y \simeq tY \lhd_B tX \simeq p_* F_K(X, Y). \tag{3.15}$$

Remark 21. Although we do not know of a reference, it should be true that a parametrized spectrum X over a connected space B with a basepoint is more or less equivalent data to a module over the ring spectrum $\Sigma^\infty_+ \Omega B$, the $\Sigma^\infty_+ \Omega B$-module corresponding to X being the fiber of X over the basepoint equipped with the holonomy action of ΩB. Under this correspondence, a parametrized spectrum over a product $B \times A$ should then be equivalent to an $\Sigma^\infty_+ \Omega B$-$\Sigma^\infty_+ \Omega A$-bimodule. From this point of view, it is not surprising that parametrized spectra over different base spaces should fit into a structure similar to the bicategory $\mathscr{M}$ of bimodules.

Remark 22. The bicategory $\mathscr{E}x$ is unsatisfactory in the sense that it fails to incorporate one essential piece of structure present in parametrized homotopy theory, namely continuous maps between base spaces. These maps should be present as kind of 1-cells, but that slot has already been taken up by parametrized spectra. Similarly, the bicategory $\mathscr{M}$ fails to capture all of the structure relevant to module theory, as it does not include ring homomorphisms as part of the structure. Michael Shulman [Shu08] has defined a more elaborate categorical structure called a framed bicategory that allows one to address these defects. However, we shall not need this more elaborate version of $\mathscr{E}x$.

3.4 Duality in closed bicategories

Our main reason for introducing the bicategories $\mathscr{E}x$ and $\mathscr{E}x_B$ was to provide a home for the duality that will give the conceptual underpinnings for the construction of umkehr maps. In this section, we will discuss the general duality theory of 1-cells in bicategories. We refer the reader to [MS06], Section 16.4 for more details.

By definition a *dual pair* (X, Y) in a bicategory $\mathscr{C}$ consists of 1-cells $X : B \to A$ and $Y : A \to B$ together with 2-cells

$$\eta : U_A \to X \odot Y$$

and

$$\varepsilon : Y \odot X \to U_B$$

such that the following diagrams commute:

$$\begin{array}{ccccc} X & \xrightarrow{\approx} & U_A \odot X & \xrightarrow{\eta \odot X} & (X \odot Y) \odot X \\ {\scriptstyle \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle \approx} \\ X & \xleftarrow[\approx]{} & X \odot U_B & \xleftarrow[X \odot \varepsilon]{} & X \odot (Y \odot X) \end{array}$$

and

$$\begin{array}{ccccc} Y & \xrightarrow{\approx} & Y \odot U_A & \xrightarrow{Y \odot \eta} & Y \odot (X \odot Y) \\ {\scriptstyle \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle \approx} \\ Y & \xleftarrow[\approx]{} & U_B \odot Y & \xleftarrow[\varepsilon \odot Y]{} & (Y \odot X) \odot Y \end{array}$$

We call X the *left dual* of Y and Y the *right dual* of X. The maps ε and η are called the *evaluation* and *coevaluation* maps, respectively. A 1-cell is called *left* or *right dualizable* if it admits a left or right dual, respectively, and the dual is then unique up to a canonical isomorphism. The definition of a dual pair may remind the reader of that of an adjoint pair of functors, and indeed an adjoint pair of functors is simply a dual pair of 1-cells in the bicategory of categories, functors and natural transformations.

If $\mathscr{C}$ is a closed bicategory and (X, Y) is a dual pair in $\mathscr{C}$ with $X : B \to A$ and $Y : A \to B$, then the adjoints of the evaluation map

$$\varepsilon : Y \odot X \to U_B$$

give isomorphisms

$$Y \xrightarrow{\approx} X \rhd U_B \quad \text{and} \quad X \xrightarrow{\approx} U_B \lhd Y$$

under which the evaluation map ε corresponds to the standard evaluation maps

$$(X \rhd U_B) \odot X \to U_B \quad \text{and} \quad Y \odot (U_B \lhd Y) \to U_B$$

of equation (3.8). Thus in a closed bicategory, every 1-cell $X : B \to A$ has a canonical candidate for a right dual, namely $X \rhd U_B$, and similarly a canonical candidate for a left dual, namely $U_A \lhd X$. However, we stress that in general neither of these 1-cells is a dual for X, as the requisite coevaluation map η may fail to exist. In general, left or right dualizability of a 1-cell X amounts to a kind of finiteness condition on X.

If $X : B \to A$ and $Y : A \to B$ and $Z : B \to C$ are 1-cells in a closed bicategory $\mathscr{C}$ and ε is a 2-cell

$$\varepsilon : Y \odot X \to U_B,$$

then the adjoint of the map

$$Z \odot Y \odot X \xrightarrow{Z \odot \varepsilon} Z \odot U_B \xrightarrow{\approx} Z$$

gives a map

$$\mu : Z \odot Y \to X \rhd Z. \tag{3.16}$$

The following proposition gives a characterization and a consequence of right dualizability.

Proposition 23. *With X, Y and ε as above, the following are equivalent:*

1. *(X, Y) is a dual pair with evaluation map ε*
2. *The map μ of equation (3.16) is an isomorphism for $Z = X$*
3. *The map μ of equation (3.16) is an isomorphism for all Z.*

Sketch of proof. Clearly (3) implies (2). Assuming (2) holds, the adjoint $U_A \to X \rhd X$ of the isomorphism $U_A \odot X \xrightarrow{\approx} X$ and the inverse of μ give a coevaluation map

$$\eta : U_A \to X \rhd X \xrightarrow{\mu^{-1}} X \odot Y$$

that pairs with the evaluation map ε to make (X, Y) into a dual pair. Finally, assuming (1) and denoting the coevaluation map by η, the map

$$X \rhd Z \xrightarrow{\approx} (X \rhd Z) \odot U_A \xrightarrow{(X \rhd Z) \odot \eta} (X \rhd Z) \odot X \odot Y \xrightarrow{\varepsilon \odot Y} Z \odot Y$$

provides an inverse for the map μ, showing that (3) holds. □

3.5 Costenoble–Waner duality

If X and Y are parametrized spectra over a space B, we may ask whether (tX, Y) or (X, tY) give dual pairs of 1-cells in $\mathscr{E}x$, and these turn out to be wholly different questions. Unrolling definitions, we see that (tX, Y) is a dual pair if there are maps

$$\Delta_! S_B \xrightarrow{\eta} \pi_1^* X \wedge_{B \times B} \pi_2^* Y \quad \text{and} \quad r_!(Y \wedge_B X) \xrightarrow{\varepsilon} S \tag{3.17}$$

satisfying the requisite identities, while (X, tY) is a dual pair if there are maps

$$S \xrightarrow{\eta} r_!(X \wedge_B Y) \quad \text{and} \quad \pi_1^* Y \wedge_{B \times B} \pi_2^* X \xrightarrow{\varepsilon} \Delta_! S_B \tag{3.18}$$

satisfying the identities. Here $\Delta : B \to B \times B$ is the diagonal map, $\pi_i : B \times B \to B$ are the projection maps, r is the unique map $r : B \to \text{pt}$, and $S = S_{\text{pt}}$ is the sphere spectrum. Taking adjoints of the η and ε in (3.17), it is now easily seen that (tX, Y) is a dual pair in $\mathscr{E}x$ if and only if X and Y are dual to each other as objects of the symmetric monoidal category $\mathrm{Ho}\,\mathscr{S}_B$. On the other hand, (X, tY) being a dual pair corresponds to a new kind of duality between spectra over B.

Definition 24. If (X, tY) as above is a dual pair, then X and Y are called *Costenoble–Waner dual* to each other. A spectrum over B is called *Costenoble–Waner dualizable* if it admits a Costenoble–Waner dual. We call a space B Costenoble–Waner dualizable if the sphere spectrum S_B over B is Costenoble–Waner dualizable.

It is easy to see that a pair of 1-cells (Z, W) is a dual pair if and only if (tW, tZ) is, so the definition is actually symmetric in X and Y. It can be shown that any spectrum X over B which is a wedge summand in $\mathrm{Ho}\,\mathscr{S}_B$ of a finite cell spectrum is Costenoble–Waner dualizable (see [MS06], Theorem 18.2.1), so in particular any finite CW complex is a Costenoble–Waner dualizable space. The following parametrized version of the Atiyah duality theorem identifies the Costenoble–Waner duals of closed manifolds.

Theorem 25 (See [MS06], Theorem 18.6.1)**.** *Suppose M is a smooth closed manifold embedded in $\mathbf{R}^L$. Then $(S_M, t\Sigma_M^{-L} S^{\nu_M})$ is a Costenoble–Waner dual pair, where ν_M denotes the normal bundle of the embedding.*

We may sometimes write $S^{-\tau_M}$ for the spectrum $\Sigma_M^{-L} S^{\nu_M}$ over M. A pleasant property of Costenoble–Waner duality is that if $f : B \to A$ is a map and (X, tY) is a Costenoble–Waner dual pair of spectra over B, then $(f_! X, t f_! Y)$ is a Costenoble–Waner dual pair of spectra over A. As Costenoble–Waner duality over the one-point space clearly reduces to ordinary Spanier–Whitehead duality of spectra, applying $r_!$ to the dual pair of the theorem we recover the classical theorem of Atiyah identifying the Spanier–Whitehead dual of $\Sigma_+^\infty M$ as $\Sigma^{-L} M^{\nu_M}$.

In addition to regular Costenoble–Waner duality, we will also need to consider relative versions of this duality defined in terms of the bicategories $\mathscr{E}x_B$ instead of $\mathscr{E}x$. In analogy with Definition 24, given a fibration $K \to B$, we call parametrized spectra X and Y over K *Costenoble–Waner B-dual* (or just *B-dual* for short) to each other if (X, tY) is a dual pair in $\mathscr{E}x_B$. A spectrum over K is called *B-dualizable* if it participates in a B-dual pair, and the space K itself is called B-dualizable if the sphere spectrum S_K is B-dualizable. Intuitively, we think of B-duality as a fiberwise version of Costenoble–Waner duality; Theorem 26 below gives this intuition precise content.

3.6 Parametrized R-modules

So far, our discussion of stable parametrized homotopy theory has been relative to the sphere spectrum S: the fibers of our parametrized spectra are S-modules. However, later on we will also need to work with parametrized R-modules, where R is a commutative S-algebra, with smash product given by fiberwise smash product over R. While the foundations developed in [MS06] are unfortunately not sufficient to support the full theory relative to R, there is little doubt that such a theory can be developed. Thus, for the remainder of the thesis, *we postulate the existence of a theory of parametrized R-modules that works in essentially same way as the theory of parametrized S-modules we have discussed.*

While not yet as fully developed as May and Sigurdsson's theory of parametrized S-modules, one approach to developing a good theory of parametrized R-modules is via ∞-categorical presheaves of R-modules. In this approach, a parametrized R-module over a space B is an ∞-functor from the ∞-groupoid determined by B to an ∞-category of (unparametrized) R-modules. Concretely, using the quasicategories of Joyal [Joy02, Lur09] to model ∞-categories, a parametrized R-module over B is a map of simplicial sets

$$\operatorname{Sing} B \to N(R\text{-mod})^\circ$$

where $\operatorname{Sing} B$ is the singular complex of B, R-mod is a simplicial model category of R-modules, $(-)^\circ$ stands for the full subcategory given by the cofibrant–fibrant objects, and N is the simplicial nerve functor. This approach is discussed in [ABG+08, ABG10], and the latter of these papers also contains a comparison with May and Sigurdsson's theory [MS06].

We will later need a classifying spectrum for (graded) R-line bundles, that is, for parametrized R-modules whose fibers are equivalent to a suspension or desuspension of R. Using the ∞-presheaf approach to the theory of parametrized R-modules, the construction of such a classifying spectrum can be sketched as follows. As before, let R-mod denote a simplicial model category of R-modules, and let $(R\text{-line}_\bullet)^\circ$ denote the subcategory whose objects are those cofibrant-fibrant R-modules which are equivalent to $\Sigma^n R$ for some $n \in \mathbf{Z}$ and whose morphisms are the equivalences between such R-modules. Then the geometric realization

$$|N(R\text{-line}_\bullet)^\circ| \tag{3.19}$$

gives a classifying space for graded R-line bundles: the isomorphism classes of graded R-line bundles over a space B are in bijective correspondence with homotopy classes of maps from B to $|N(R\text{-line}_\bullet)^\circ|$. Moreover, the smash product (over R) of R-modules gives the ∞-category $N(R\text{-line}_\bullet)^\circ$ symmetric monoidal structure, which makes the space $|N(R\text{-line}_\bullet)^\circ|$ a group-like E_∞ space, and hence the zeroth space in a connective spectrum. We will denote this spectrum by $\text{line}_\bullet(R)$. This is spectrum is then our desired classifying spectrum: isomorphism classes of graded R-lines over B are in bijective correspondence with homotopy classes of maps of spectra from $\Sigma^\infty_+ B$ to $\text{line}_\bullet(R)$. We note that the zeroth space (3.19) of $\text{line}_\bullet(R)$ is equivalent to

$$\mathbf{Z}/l \times |N(R\text{-line})^\circ| \simeq \mathbf{Z}/l \times BGL_1(R) \tag{3.20}$$

where l is the period of R or 0 if R is not periodic. Here $(R\text{-line})^\circ$ is defined in the same way as $(R\text{-line}_\bullet)^\circ$ except that we require all objects to be equivalent to $\Sigma^0 R$. The equivalence in (3.20) follows from [ABG$^+$08].

Given a finite-dimensional vector space V, we can associate to V the E-module $S^V \wedge E$, where S^V denotes the one-point compactification of V. This association gives rise to a map

$$ko \to \text{line}_\bullet(R)$$

where ko is the connective real K-theory spectrum. This map will feature later in Section 5.4 in the definition of universal R-orientation, Definition 37.

Chapter 4

Umkehr maps in the twisted setting

In this chapter, we discuss umkehr maps (or wrong-way maps) in the parametrized setting. The first section is devoted to the proof of Theorem 26, the main result and workhorse of the chapter. The theorem is a slight partial generalization—from fiber bundles to fibrations, but only non-equivariantly—of May and Sigurdsson's [MS06] Theorem 19.5.2, which they call the Fiberwise Costenoble–Waner Duality Theorem. (A result similar to May and Sigurdsson's was proven earlier by Po Hu; see [Hu03], Theorem 4.9). The second section then explains how Theorem 26 gives rise to umkehr maps. The umkehr maps we will need in the sequel are pretransfer-type maps, with history (at least in the untwisted case) going back to Becker and Gottlieb's original paper on transfer maps [BG75]. However, as we will see, Theorem 26 naturally gives rise to more general umkehr maps as well.

4.1 The fiberwise Costenoble–Waner duality theorem for fibrations

Our goal in this section is to prove the following version of the Fiberwise Costenoble–Waner Duality Theorem, [MS06], Theorem 19.5.2. The result is a generalization of

May and Sigurdsson's theorem, which is deals with fiber bundles instead of fibrations.

Theorem 26. *Let $p : E \to B$ be a fibration whose fibers are Costenoble–Waner dualizable. Then E is Costenoble–Waner B-dualizable, and for any Costenoble–Waner B-dual T_p of S_E, we have a natural equivalence*

$$p_!(X \wedge_B T_p) \simeq p_* X \tag{4.1}$$

for $X \in \operatorname{Ho} \mathscr{S}_E$.

The proof of the theorem hinges on relating duality in the bicategory $\mathscr{E}x_B$ to duality in the bicategory $\mathscr{E}x$. Let us start by introducing the relevant categorical notions. If $\mathscr{C}$ and $\mathscr{D}$ are bicategories, a *pseudofunctor*

$$F : \mathscr{C} \to \mathscr{D}$$

consists of, first, a map from the 0-cells of $\mathscr{C}$ to the 0-cells of $\mathscr{D}$; second, for each pair A, B of 0-cells of $\mathscr{C}$, a functor

$$F : \mathscr{C}(A, B) \to \mathscr{D}(FA, FB);$$

and third, coherent natural isomorphisms

$$\beta : FY \odot FX \xrightarrow{\approx} F(Y \odot X) \quad \text{and} \quad \kappa : U_{FA} \xrightarrow{\approx} FU_A. \tag{4.2}$$

If $\mathscr{C}$ and $\mathscr{D}$ are closed bicategories, a pseudofunctor $F : \mathscr{C} \to \mathscr{D}$ is called *closed* if the adjoints

$$\alpha : F(X \rhd Z) \to F(X) \rhd F(Z) \tag{4.3}$$

of the maps

$$F(X \rhd Z) \odot F(X) \xrightarrow{\beta} F((X \rhd Z) \odot X) \xrightarrow{F\varepsilon} F(Z)$$

are isomorphisms.

If $f : A \to B$ is a map, then according to [MS06], Proposition 19.3.4, f induces a pseudofunctor $f^* : \mathscr{E}x_B \to \mathscr{E}x_A$ in the following way: On objects, f^* sends a 0-cell $K \to B$ to the pullback $f^*K \to A$, as in the square

$$\begin{array}{ccc} f^*K & \longrightarrow & K \\ \big\downarrow & & \big\downarrow \\ A & \xrightarrow{\;f\;} & B \end{array}$$

and for 0-cells $K \to B$ and $L \to B$ of $\mathscr{E}x_B$, the functor

$$f^* : \mathscr{E}x_B(K, L) \to \mathscr{E}x_A(f^*K, f^*L)$$

is defined by pull-back along the map

$$f^*K \times_A f^*L \approx f^*(K \times_B L) \to K \times_B L.$$

If X and Y are as in equation (3.12), the natural isomorphism β of (4.2) is given by the composite

$$\begin{aligned} f^*Y \odot_A f^*X &\simeq (\pi_{\overline{MK}})_! \left((\pi_{\overline{ML}})^* f^*Y \wedge_{\overline{MLK}} (\pi_{\overline{LK}})^* f^*X\right) \\ &\xrightarrow{\simeq} (\pi_{\overline{MK}})_! \left(f^*(\pi_{ML})^*Y \wedge_{\overline{MLK}} f^*(\pi_{LK})^*X\right) \\ &\xrightarrow{\simeq} (\pi_{\overline{MK}})_! f^* \left((\pi_{ML})^*Y \wedge_{MLK} (\pi_{LK})^*X\right) \\ &\xrightarrow{\simeq} f^*(\pi_{MK})_! \left((\pi_{ML})^*Y \wedge_{MLK} (\pi_{LK})^*X\right) \\ &\simeq f^*(Y \odot_B X) \end{aligned} \tag{4.4}$$

where we have used overline to denote pullback along f and we have omitted $\times_B$ and $\times_A$ from the notation. Here the maps denoted by π are projections domain $M \times_B L \times_B K$ or $f^*M \times_A f^*L \times_A f^*K$ and with the indicated codomains, and the next-to-last equivalence is given by the analogue of Proposition 19 for the homotopy category of parametrized spectra. As the precise definition of the natural isomorphism κ of (4.2) will not be important to us, we shall not elaborate on it here.

As one might expect, the pseudofunctor $f^* : \mathscr{E}x_B \to \mathscr{E}x_A$ is closed, and proving this will be our next goal. To make the proof more conceptual, we will first indulge in some more category theory. Suppose

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

and

$$\bar{L} : \bar{\mathcal{C}} \rightleftarrows \bar{\mathcal{D}} : \bar{R}$$

are adjunctions and $B : \mathcal{C} \to \bar{\mathcal{C}}$ and $B' : \mathcal{D} \to \bar{\mathcal{D}}$ are functors. Then two natural transformations

$$\sigma : \bar{L}B \to B'L \quad \text{and} \quad \tau : BR \to \bar{R}B'$$

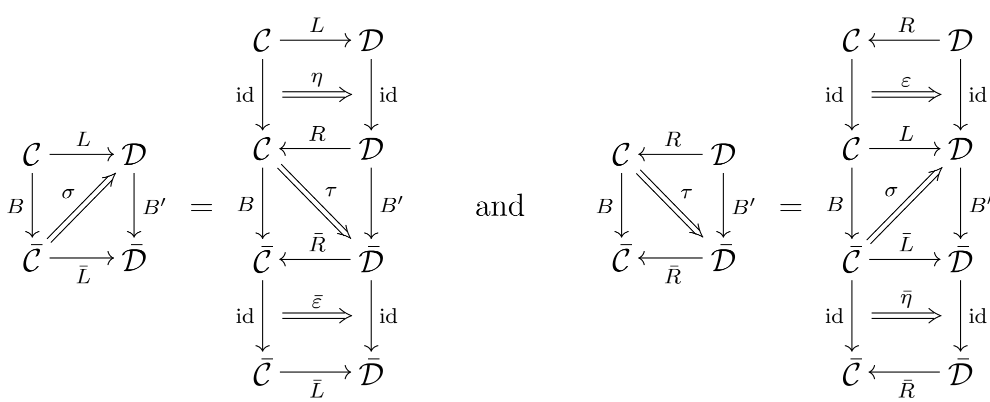
are called *mates* when σ is the composite

$$\bar{L}B \xrightarrow{\bar{L}B\eta} \bar{L}BRL \xrightarrow{\bar{L}\tau L} \bar{L}\bar{R}B'L \xrightarrow{\bar{\varepsilon}B'L} B'L$$

or equivalently when τ is the composite

$$BR \xrightarrow{\bar{\eta}BR} \bar{R}\bar{L}BR \xrightarrow{\bar{R}\sigma R} \bar{R}B'LR \xrightarrow{\bar{R}B'\varepsilon} \bar{R}B'.$$

Here η, $\bar{\eta}$ and ε, $\bar{\varepsilon}$ denote the units and the counits of the adjunctions, respectively. As the domain and codomain of σ feature the left adjoints L and $\bar{L}$, we sometimes call σ the *left mate* of τ, and similarly τ the *right mate* of σ. Pictorially, σ and τ are mates when



Recalling that the adjoint of a morphism $f : \bar{L}C \to D$ under the adjunction

$$\bar{L} : \bar{\mathcal{C}} \rightleftarrows \bar{\mathcal{D}} : \bar{R}$$

is the composite

$$C \xrightarrow{\bar{\eta}} \bar{R}\bar{L}(C) \xrightarrow{\bar{R}f} \bar{R}D,$$

we see that the right mate of

$$\sigma : \bar{L}B \to B'L$$

is precisely the adjoint of

$$\bar{L}BR \xrightarrow{\sigma R} B'LR \xrightarrow{B'\varepsilon} B'.$$

In particular, taking (L, R) and $(\bar{L}, \bar{R})$ to be $(-\odot X, X \rhd -)$ and $(-\odot F(X), F(X) \rhd -)$, respectively, and letting B and B' be given by F, we have the following lemma.

Lemma 27. *Suppose $F : \mathscr{C} \to \mathscr{D}$ is a pseudofunctor. Then for any 1-cell X the natural transformation*

$$\alpha : F(X \rhd -) \to F(X) \rhd F(-)$$

of (4.3) *is the right mate of the natural isomorphism*

$$\beta : F(-) \odot F(X) \xrightarrow{\approx} F(- \odot X)$$

of (4.2). □

We are now ready to prove that f^* is closed.

Proposition 28. *Suppose $f : A \to B$ is a continuous map. Then the pseudofunctor*

$$f^* : \mathscr{E}x_B \to \mathscr{E}x_A$$

is closed.

Proof. Let $K \to B$, $L \to B$ and $M \to B$ be 0-cells and let $X : K \to L$ be a 1-cell in

$\mathscr{E}x_B$. The adjunctions

$$(-) \odot_B X : \mathscr{E}x_B(L, M) \rightleftarrows \mathscr{E}x_B(K, M) : X \rhd_B (-)$$

and

$$(-) \odot_A f^*(X) : \mathscr{E}x_B(f^*L, f^*M) \rightleftarrows \mathscr{E}x_B(f^*K, f^*M) : f^*X \rhd_A (-)$$

split as composites

$$\operatorname{Ho}\mathscr{S}_{ML} \underset{(\pi_{ML})_*}{\overset{(\pi_{ML})^*}{\rightleftarrows}} \operatorname{Ho}\mathscr{S}_{MLK} \underset{F_{MLK}((\pi_{LK})^*X,-)}{\overset{(-)\wedge_{MLK}(\pi_{LK})^*X}{\rightleftarrows}} \operatorname{Ho}\mathscr{S}_{MLK} \underset{(\pi_{MK})^*}{\overset{(\pi_{MK})_!}{\rightleftarrows}} \operatorname{Ho}\mathscr{S}_{MK}$$

and

$$\operatorname{Ho}\mathscr{S}_{\overline{ML}} \underset{(\pi_{\overline{ML}})_*}{\overset{(\pi_{\overline{ML}})^*}{\rightleftarrows}} \operatorname{Ho}\mathscr{S}_{\overline{MLK}} \underset{F_{\overline{MLK}}((\pi_{\overline{LK}})^*f^*X,-)}{\overset{(-)\wedge_{\overline{MLK}}(\pi_{\overline{LK}})^*f^*X}{\rightleftarrows}} \operatorname{Ho}\mathscr{S}_{\overline{MLK}} \underset{(\pi_{\overline{MK}})^*}{\overset{(\pi_{\overline{MK}})_!}{\rightleftarrows}} \operatorname{Ho}\mathscr{S}_{\overline{MK}}$$

where again we have used overline to denote pullback along f and we have omitted $\times_B$ and $\times_A$ from the notation. From equation (4.4), we see that the natural isomorphism

$$\beta : f^*(-) \odot_A f^*(X) \xrightarrow{\simeq} f^*(- \odot_B X)$$

is the composite 2-cell in

$$\begin{array}{ccccccc}
\operatorname{Ho}\mathscr{S}_{ML} & \xrightarrow{(\pi_{ML})^*} & \operatorname{Ho}\mathscr{S}_{MLK} & \xrightarrow{(-)\wedge_{MLK}(\pi_{LK})^*X} & \operatorname{Ho}\mathscr{S}_{MLK} & \xrightarrow{(\pi_{MK})_!} & \operatorname{Ho}\mathscr{S}_{MK} \\
\downarrow f^* & \overset{\beta_1}{\nearrow\!\!\!\nearrow} & \downarrow f^* & \overset{\beta_2}{\nearrow\!\!\!\nearrow} & \downarrow f^* & \overset{\beta_3}{\nearrow\!\!\!\nearrow} & \downarrow f^* \\
\operatorname{Ho}\mathscr{S}_{\overline{ML}} & \xrightarrow[(\pi_{\overline{ML}})^*]{} & \operatorname{Ho}\mathscr{S}_{\overline{MLK}} & \xrightarrow[(-)\wedge_{\overline{MLK}}(\pi_{\overline{LK}})^*f^*X]{} & \operatorname{Ho}\mathscr{S}_{\overline{MLK}} & \xrightarrow[(\pi_{\overline{MK}})_!]{} & \operatorname{Ho}\mathscr{S}_{\overline{MK}}
\end{array}$$

where β_1 is the isomorphism

$$\beta_1 : (\pi_{\overline{ML}})^* f^* \xrightarrow{\simeq} f^*(\pi_{ML})^*,$$

β_2 is the composite

$$\begin{aligned}\beta_2 : f^*(-) \wedge_{\overline{MLK}} (\pi_{\overline{LK}})^* f^* X &\xrightarrow{\simeq} f^*(-) \wedge_{\overline{MLK}} f^*(\pi_{LK})^* X \\ &\xrightarrow{\simeq} f^*\left((-) \wedge_{MLK} (\pi_{LK})^* X\right)\end{aligned}$$

and β_3 is the isomorphism

$$\beta_3 : (\pi_{\overline{MK}})_! f^* \xrightarrow{\simeq} f^*(\pi_{MK})_!$$

given by the analogue of Proposition 19 for the homotopy categories of parametrized spectra. Now the right mate

$$\alpha : f^*(X \rhd_B -) \to f^*(X) \rhd_A f^*(-)$$

of β is the composite 2-cell

$$\begin{array}{ccccccc}
\operatorname{Ho}\mathscr{S}_{ML} & \xleftarrow{(\pi_{ML})_*} & \operatorname{Ho}\mathscr{S}_{MLK} & \xleftarrow{F_{MLK}((\pi_{LK})^*X,-)} & \operatorname{Ho}\mathscr{S}_{MLK} & \xleftarrow{(\pi_{MK})^*} & \operatorname{Ho}\mathscr{S}_{MK} \\
{\scriptstyle \mathrm{id}}\downarrow & \overset{\varepsilon}{\Longrightarrow} & {\scriptstyle \mathrm{id}}\downarrow & \overset{\varepsilon}{\Longrightarrow} & {\scriptstyle \mathrm{id}}\downarrow & \overset{\varepsilon}{\Longrightarrow} & \downarrow{\scriptstyle \mathrm{id}} \\
\operatorname{Ho}\mathscr{S}_{ML} & \xrightarrow{(\pi_{ML})^*} & \operatorname{Ho}\mathscr{S}_{MLK} & \xrightarrow{(-)\wedge_{MLK}(\pi_{LK})^*X} & \operatorname{Ho}\mathscr{S}_{MLK} & \xrightarrow{(\pi_{MK})_!} & \operatorname{Ho}\mathscr{S}_{MK} \\
{\scriptstyle f^*}\downarrow & \overset{\beta_1}{\nearrow\!\!\!\nearrow} & {\scriptstyle f^*}\downarrow & \overset{\beta_2}{\nearrow\!\!\!\nearrow} & {\scriptstyle f^*}\downarrow & \overset{\beta_3}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle f^*} \\
\operatorname{Ho}\mathscr{S}_{\overline{ML}} & \xrightarrow[(\pi_{\overline{ML}})^*]{} & \operatorname{Ho}\mathscr{S}_{\overline{MLK}} & \xrightarrow[(-)\wedge_{\overline{MLK}}(\pi_{\overline{LK}})^*f^*X]{} & \operatorname{Ho}\mathscr{S}_{\overline{MLK}} & \xrightarrow[(\pi_{\overline{MK}})_!]{} & \operatorname{Ho}\mathscr{S}_{\overline{MK}} \\
{\scriptstyle \mathrm{id}}\downarrow & \overset{\eta}{\Longrightarrow} & {\scriptstyle \mathrm{id}}\downarrow & \overset{\eta}{\Longrightarrow} & {\scriptstyle \mathrm{id}}\downarrow & \overset{\eta}{\Longrightarrow} & \downarrow{\scriptstyle \mathrm{id}} \\
\operatorname{Ho}\mathscr{S}_{\overline{ML}} & \xleftarrow[(\pi_{\overline{ML}})_*]{} & \operatorname{Ho}\mathscr{S}_{\overline{MLK}} & \xleftarrow[F_{\overline{MLK}}((\pi_{\overline{LK}})^*f^*X,-)]{} & \operatorname{Ho}\mathscr{S}_{\overline{MLK}} & \xleftarrow[(\pi_{\overline{MK}})^*]{} & \operatorname{Ho}\mathscr{S}_{\overline{MK}}
\end{array}$$

But here the composite of the 2-cells in the left column is the right mate of β_1, which is the natural isomorphism

$$f^*(\pi_{ML})_* \xrightarrow{\simeq} (\pi_{\overline{ML}})_* f^*$$

of the analogue of Proposition 19; the composite of the 2-cells in the middle column is the right mate of β_2, which is the natural isomorphism

$$\begin{aligned} f^* F_{MLK}((\pi_{LK})^* X, -) &\xrightarrow{\simeq} F_{\overline{MLK}}(f^*(\pi_{LK})^* X, f^*(-)) \\ &\xrightarrow{\simeq} F_{\overline{MLK}}((\pi_{\overline{LK}})^* f^* X, f^*(-)); \end{aligned}$$

and the composite of the 2-cells in the right column is the natural isomorphism

$$f^*(\pi_{MK})^* \xrightarrow{\simeq} (\pi_{\overline{MK}})^* f^*.$$

It follows that α is a natural isomorphism as well, as desired. □

Suppose $X : A \to B$, $Y : A \to C$ and $Z : C \to D$ are 1-cells in a closed bicategory $\mathscr{C}$. Then the adjoint of the map

$$Z \odot (X \rhd Y) \odot X \xrightarrow{Z \odot \varepsilon} Z \odot Y$$

gives a map

$$\mu_{XYZ} : Z \odot (X \rhd Y) \to X \rhd (Z \odot Y).$$

Let us denote by μ_{XZ} the composite

$$\mu_{XZ} : Z \odot (X \rhd U_A) \xrightarrow{\mu_{XU_AZ}} X \rhd (Z \odot U_A) \xrightarrow{\approx} X \rhd Z.$$

Then μ_{XZ} is simply the map μ of equation (3.16) associated with the standard evaluation map

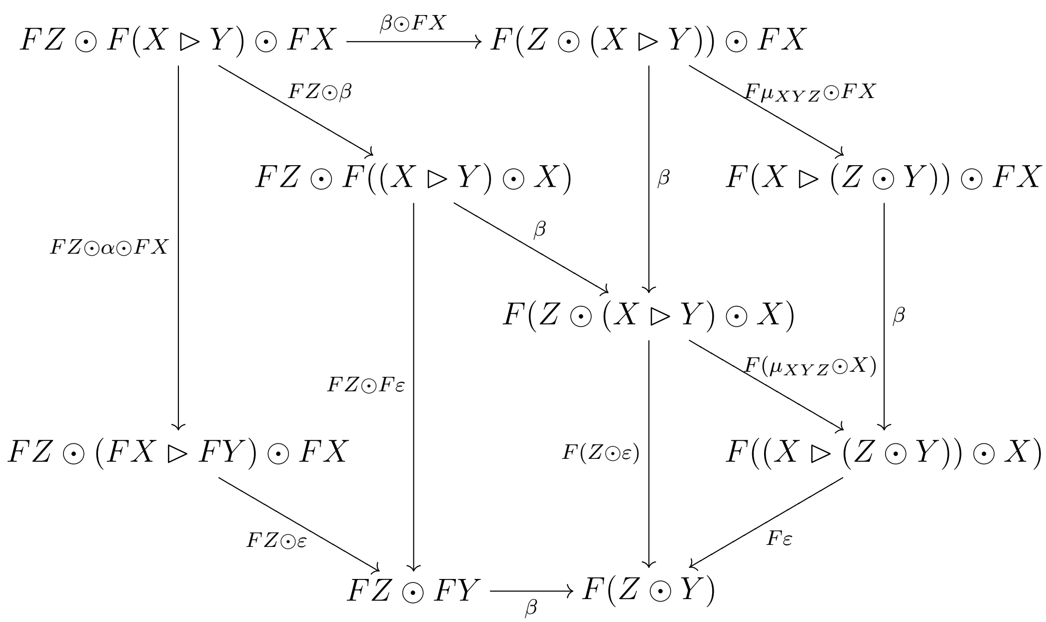
$$\varepsilon : (X \rhd U_A) \odot X \to U_A.$$

Our next goal is to show that $F\mu_{XZ}$ is isomorphic to $\mu_{FX,FZ}$ when F is a closed pseudofunctor. We will start with the following lemma, which spells out the relationship between $F\mu_{XYZ}$ and $\mu_{FX,FY,FZ}$.

Lemma 29. *Suppose $F : \mathscr{C} \to \mathscr{D}$ is a pseudofunctor between closed bicategories. Then the following diagram commutes:*

$$\begin{array}{ccc}
F(Z \odot (X \rhd Y)) & \xrightarrow{F\mu_{XYZ}} & F(X \rhd (Z \odot Y)) \\
\uparrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} \\
FZ \odot F(X \rhd Y) & & FX \rhd F(Z \odot Y) \\
\downarrow{\scriptstyle FZ\odot\alpha} & & \uparrow{\scriptstyle FX \rhd \beta} \\
FZ \odot (FX \rhd FY) & \xrightarrow[\mu_{FX,FY,FZ}]{} & FX \rhd (FZ \odot FY)
\end{array} \tag{4.5}$$

Proof. We have the following diagram:



(4.6)

Here the triangle on top commutes by a coherence condition required of β, and the parallelogram on the right and the trapezoid in the middle commute by naturality. The parallelogram on the left commutes by the definition of α as an adjoint of $F\varepsilon \circ \beta$, and the triangle in the bottom right hand corner commutes by the definition of μ_{XYZ} as an adjoint of $Z \odot \varepsilon$. Now the composite

$$\alpha \circ F\mu_{XYZ} \circ \beta$$

in diagram (4.5) is adjoint to the composition of the maps along the top- and right-edges of diagram (4.6), while the other composite in diagram (4.5) is adjoint to the composite of the maps along the left and bottom outer edge of diagram (4.6). The claim follows. □

We are now ready to prove that $F\mu_{XY}$ and $\mu_{FX,FY}$ are isomorphic.

Lemma 30. *Suppose $F : \mathscr{C} \to \mathscr{D}$ is a closed pseudofunctor. Then the maps*

$$F(Z \odot (X \rhd U_A)) \xrightarrow{F\mu_{XZ}} F(X \rhd Z)$$

and

$$F(Z) \odot (FX \rhd U_{FA}) \xrightarrow{\mu_{FX,FZ}} FX \rhd FZ$$

are related through a zigzag of coherence isomorphisms.

Proof. We have the diagram

$$\begin{array}{ccccc}
F(Z \odot (X \rhd U_A)) & \xrightarrow{F\mu_{XU_AZ}} & F(X \rhd (Z \odot U_A)) & \xrightarrow{\approx} & F(X \rhd Z) \\
\beta \uparrow \approx & & \approx \downarrow \alpha & & \\
FZ \odot F(X \rhd U_A) & & FX \rhd F(Z \odot U_A) & & \approx \downarrow \alpha \\
\alpha \downarrow \approx & & \approx \uparrow \beta & \searrow \approx & \\
FZ \odot (FX \rhd FU_A) & \xrightarrow{\mu_{FX,FU_A,FZ}} & FX \rhd (FZ \odot FU_A) & & \\
\kappa \uparrow \approx & & \approx \uparrow \kappa & & \\
FZ \odot (FX \rhd U_{FA}) & \xrightarrow{\mu_{FX,U_{FA},FZ}} & FX \rhd (FZ \odot U_{FA}) & \xrightarrow{\approx} & FX \rhd FZ
\end{array}$$

The rectangle in the top left hand corner commutes by Lemma 29, while the rectangle in the bottom left hand corner and the trapezoid in the top right hand corner commute by naturality. Finally, the triangle commutes by a coherence property required of pseudofunctors. As $F\mu_{XZ}$ is the composite of the maps in the top row and $\mu_{FX,FZ}$ is the composite of the maps in the bottom row of the diagram, the claim follows. □

The proof of Theorem 26 is now easy.

Proof of Theorem 26. We will show that the standard evaluation map

$$(S_E \rhd_B U_E) \odot_B S_E \xrightarrow{\varepsilon} U_E$$

makes $(S_E, S_E \rhd_B U_E)$ a Costenoble–Waner B-dual pair. By Proposition 23, it is enough to show that the map

$$\mu_{S_E,S_E} : S_E \odot_B (S_E \rhd_B U_E) \to S_E \rhd_B S_E$$

is an equivalence in $\mathscr{E}x_B(B,B) = \mathrm{Ho}\,\mathscr{S}_B$, and by the analogue of Proposition 20 for parametrized spectra, μ_{S_E,S_E} is an equivalence if the map

$$b^*(\mu_{S_E,S_E}) : b^*(S_E \odot_B (S_E \rhd_B U_E)) \to b^*(S_E \rhd_B S_E)$$

is an equivalence in $\mathscr{E}x(\mathrm{pt},\mathrm{pt}) = \mathscr{S}_{\mathrm{pt}}$ for every $b \in B$. By Proposition 28, $b^* : \mathscr{E}x_B \to \mathscr{E}x$ is a closed pseudofunctor, so by Lemma 30 the map $b^*(\mu_{S_E,S_E})$ is equivalent to the map

$$\mu_{b^*S_E,b^*S_E} : b^*S_E \odot (b^*S_E \rhd U_{b^*E}) \to b^*S_E \rhd b^*S_E.$$

As b^*E is just the fiber E_b of E over b, and b^*S_E is equivalent to S_{E_b} over E_b, the map $\mu_{b^*S_E,b^*S_E}$ is equivalent to the map

$$\mu_{S_{E_b},S_{E_b}} : S_{E_b} \odot (S_{E_b} \rhd U_{E_b}) \to S_{E_b} \rhd S_{E_b}.$$

But this last map is an equivalence by Proposition 23 and the assumption that the fiber E_b is Costenoble–Waner dualizable. Thus $b^*(\mu_{S_E,S_E})$ is also an equivalence, and S_E is Costenoble–Waner B-dualizable, as claimed.

Suppose now T_p is a Costenoble–Waner B-dual for S_E. By Proposition 23, we then have a natural equivalence

$$\mu : X \odot_B T_p \xrightarrow{\simeq} S_E \rhd_B X$$

for all spectra X over E. By equation (3.15), the domain of μ is equivalent to $p_!(X \wedge_E T_p)$, while the codomain of μ is equivalent to $p_* F_E(S_E, X) \simeq p_* X$. Thus the desired comparison (4.1) follows. □

4.2 Umkehr maps

In this section, we explain how Theorem 26 can be used to construct umkehr maps in the twisted setting. Throughout the section we will work on the level of homotopy categories.

Let us start by introducing notation. Suppose B is a space and X is a spectrum over B. Then we denote

$$H_\bullet(B;\, X) = r_! X \quad \text{and} \quad H^\bullet(B;\, X) = r_* X \tag{4.7}$$

where r is the unique map from B to pt. The notation is supposed to suggest homology and cohomology with twisted coefficients: If X is, say, an R-line bundle over B, then the homotopy groups of $H_\bullet(B;\, X) = r_! X$ and $H^\bullet(B;\, X) = r_* X$ are the twisted R-homology and R-cohomology groups of B, with the twisting given by the R-line bundle X. The case of a trivial bundle corresponds to the situation where there is no twisting. If X is the trivial spectrum over B with fiber the spectrum F, then

$$\pi_q H_\bullet(B;\, X) \approx F_q(B) \quad \text{and} \quad \pi_q H^\bullet(B;\, X) \approx F^{-q}(B).$$

It is convenient to generalize the notation (4.7) to a parametrized setting. Suppose $p : E \to B$ is a fibration, and let X be a spectrum over E. Then we denote

$$\mathcal{H}_\bullet^{(B)}(E;\, X) = p_! X \quad \text{and} \quad \mathcal{H}^\bullet_{(B)}(E;\, X) = p_* X,$$

sometimes dropping (B) from the notation when the base space B is clear from the context. We think of $\mathcal{H}_\bullet^{(B)}$ and $\mathcal{H}^\bullet_{(B)}$ as fiberwise versions of $H_\bullet$ and $H^\bullet$, respectively:

if $b \in B$, then

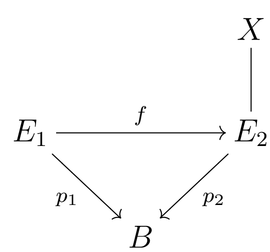
$$\mathcal{H}_\bullet^{(B)}(E;\, X)_b \simeq H_\bullet(E_b;\, i_b^* X) \quad \text{and} \quad \mathcal{H}^\bullet_{(B)}(E;\, X)_b \simeq H^\bullet(E_b;\, i_b^* X)$$

where $(-)_b$ denotes the fiber over b and $i_b : E_b \hookrightarrow E$ is the inclusion. We note that we have

$$H_\bullet(E;\, X) \simeq r_! \mathcal{H}_\bullet^{(B)}(E;\, X) \quad \text{and} \quad H^\bullet(E;\, X) \simeq r_* \mathcal{H}^\bullet_{(B)}(E;\, X)$$

where r continues to denote the map from B to the one-point space.

Given a commutative diagram



(4.8)

where p_1 and p_2 are fibrations and X is a spectrum over E_2, we define

$$f_\sharp : \mathcal{H}_\bullet^{(B)}(E_1;\, f^* X) \to \mathcal{H}_\bullet^{(B)}(E_2;\, X) \tag{4.9}$$

to be the composite

$$\mathcal{H}_\bullet^{(B)}(E_1;\, f^* X) = (p_1)_! f^* X \simeq (p_2)_! f_! f^* X \to (p_2)_! X = \mathcal{H}_\bullet^{(B)}(E_2;\, X)$$

where the equivalence follows from the factorization $p_1 = p_2 f$ and the arrow is given by the counit of the adjoint pair $(f_!, f^*)$. Similarly, we define

$$f^\sharp : \mathcal{H}^\bullet_{(B)}(E_2;\, X) \to \mathcal{H}^\bullet_{(B)}(E_1;\, f^* X) \tag{4.10}$$

as the composite

$$\mathcal{H}^\bullet_{(B)}(E_2;\, X) = (p_2)_* X \to (p_2)_* f_* f^* X \simeq (p_1)_* f^* X = \mathcal{H}^\bullet_{(B)}(E_1;\, f^* X)$$

where the arrow is now given by the unit of the adjoint pair (f^*, f_*). Applying $r_!$ to (4.9) and r_* to (4.10), we also obtain induced maps

$$f_\sharp : H_\bullet(E_1;\, f^*X) \to H_\bullet(E_2;\, X)$$

and

$$f^\sharp : H^\bullet(E_2;\, X) \to H^\bullet(E_1;\, f^*X).$$

We choose not to distinguish these maps notationally from the previous ones.

Suppose now that $p : E \to B$ is a fibration with Costenoble–Waner dualizable fibers. Then Theorem 26 gives an equivalence

$$\mathcal{H}_\bullet^{(B)}(E;\, X \wedge_E T_p) \simeq \mathcal{H}^\bullet_{(B)}(E;\, X) \tag{4.11}$$

where X is any spectrum over E and T_p is a Costenoble–Waner B-dual of E. Assuming that the fibers of p_1 and p_2 are Costenoble–Waner dualizable, the equivalence (4.11) allows us to associate various umkehr maps to f. Denoting by T_1 and T_2 the Costenoble–Waner B-duals of E_1 and E_2, respectively, we define a map

$$f^\leftarrow : \mathcal{H}_\bullet(E_2;\, X \wedge_{E_2} T_2) \to \mathcal{H}_\bullet(E_1;\, f^*X \wedge_{E_1} T_1) \tag{4.12}$$

as the composite

$$\mathcal{H}_\bullet(E_2;\, X \wedge_{E_2} T_2) \simeq \mathcal{H}^\bullet(E_2;\, X) \xrightarrow{f^\sharp} \mathcal{H}^\bullet(E_1;\, f^*X) \simeq \mathcal{H}_\bullet(E_1;\, f^*X \wedge_{E_1} T_1).$$

Assuming that T_2 is invertible in the sense that there exists a spectrum T_2^{-1} over E_2 such that $T_2 \wedge_{E_2} T_2^{-1} \simeq S_{E_2}$, substituting $X \wedge_{E_2} T_2^{-1}$ for X in (4.12), we also obtain an umkehr map

$$\mathcal{H}_\bullet(E_2;\, X) \to \mathcal{H}_\bullet(E_1;\, f^*X \wedge_{E_1} (f^*(T_2)^{-1} \wedge_{E_1} T_1)) \tag{4.13}$$

which we continue to denote by $f^\leftarrow$. Similarly, we can use the equivalence (4.11) to define umkehr maps in cohomology. Assuming that T_1 and T_2 are both invertible, the composite

$$\mathcal{H}^\bullet(E_1;\, f^*X \wedge_{E_1} T_1^{-1}) \simeq \mathcal{H}_\bullet(E_1;\, f^*X) \xrightarrow{f_\sharp} \mathcal{H}_\bullet(E_2;\, X) \simeq \mathcal{H}^\bullet(E_2;\, X \wedge_{E_2} T_2^{-1})$$

gives us an umkehr map

$$f_\leftarrow : \mathcal{H}^\bullet(E_1;\, f^*X \wedge_{E_1} T_1^{-1}) \to \mathcal{H}^\bullet(E_2;\, X \wedge_{E_2} T_2^{-1}) \tag{4.14}$$

and substituting $X \wedge_{E_2} T_2$ for X we also get an umkehr map

$$f_\leftarrow : \mathcal{H}^\bullet(E_1;\, f^*X \wedge_{E_1} (f^*T_2 \wedge_{E_1} T_1^{-1})) \to \mathcal{H}^\bullet(E_2;\, X); \tag{4.15}$$

in fact, for the definition of this latter umkehr map, it is enough to assume that just T_1 is invertible. Again, applying $r_!$ to the homological umkehr maps and r^* to the cohomological ones, we obtain umkehr maps featuring $H_\bullet$ and $H^\bullet$ instead of $\mathcal{H}_\bullet$ and $\mathcal{H}^\bullet$, respectively. We will use the notation $f^\leftarrow$ and $f_\leftarrow$ for these maps as well.

In general, special assumptions are necessary to guarantee that the twisting T_p associated with a fibration

$$p : E \to B$$

is invertible. A common situation where T_p is invertible and easy to understand is when p is a fiber bundle whose fibers are closed manifolds; in that case, it follows from Theorem 18.6.1, Proposition 18.3.2 and Corollary 19.4.4 of [MS06] that $T_p \simeq S_E^{-\tau_{\mathrm{vert}}}$, where τ_{vert} is the vertical tangent bundle of p. More generally, T_p is invertible when p is a fibration whose fibers are homotopy equivalent to closed manifolds.

We now turn to examples, focusing on homology. For our purposes, the most important umkehr maps correspond to the case where $E_2 = B$ and p_2 is the identity map in diagram (4.8). Writing E for E_1, p for p_1 and f and T_p for T_1, the umkehr map (4.12) becomes

$$p^\leftarrow : \mathcal{H}_\bullet(B;\, X) \to \mathcal{H}_\bullet(E;\, p^*X \wedge_E T_p).$$

We call this map (or the one obtained from it by applying $r_!$) the *pretransfer* map. To motivate the terminology, consider the case where $p : E \to B$ is a fiber bundle whose fibers are closed manifolds, so that $T_p \simeq S_E^{-\tau_{\mathrm{vert}}}$. Then, if X is a trivial bundle of spectra with fiber the spectrum F, applying $r_!$ to the above map and taking homotopy groups gives us the factor

$$F_*(B) \to \tilde{F}_*(E^{-\tau_{\mathrm{vert}}})$$

of the full Becker–Gottlieb transfer map

$$F_*(B) \to F_*(E)$$

of [BG75].

While the pretransfer maps discussed above are the umkehr maps that will be the most important for us in the sequel, we would like to point out that the case where the space B in diagram (4.8) is a one-point space is also of interest. In that case, assuming that E is a closed manifold M and using Theorem 25 to identify $S_M^{-\tau_M}$ as the Costenoble–Waner dual of M, the equivalence (4.11) becomes

$$H_\bullet(M;\, X \wedge_M S_M^{-\tau_M}) \simeq H^\bullet(M;\, X), \tag{4.16}$$

a strong form of unoriented Poincaré duality. If $f : M_1 \to M_2$ is a map between closed manifolds, then (4.12) becomes a map

$$f^{\leftarrow} : H_\bullet(M_2;\, X \wedge_{M_2} S_{M_2}^{-\tau_{M_2}}) \to H_\bullet(M_1;\, f^*X \wedge_{M_1} S_{M_1}^{-\tau_{M_1}}) \tag{4.17}$$

and (4.13) a map

$$\begin{aligned} H_\bullet(M_2;\, X) \to & H_\bullet(M_1;\, f^*X \wedge_{M_1} (f^*(S_{M_2}^{-\tau_{M_2}})^{-1} \wedge_{M_1} S_{M_1}^{-\tau_{M_1}})) \\ \simeq & H_\bullet(M_1;\, f^*X \wedge_{M_1} S_{M_1}^{\nu_f}) \end{aligned} \tag{4.18}$$

where

$$\nu_f = f^*\tau_{M_2} - \tau_{M_1}$$

is the virtual normal bundle of f. As pointed out by Cohen and Klein [CK09], taking

f to be the diagonal map $\Delta : M \to M \times M$ of a closed manifold M and taking X to be $(LM \times LM, \mathrm{ev} \times \mathrm{ev})_+$, where $\mathrm{ev} : LM \to M$ is evaluation at the basepoint, we obtain an umkehr map relevant to string topology: with these choices, (4.17) becomes the map

$$\begin{aligned} \Delta^{\leftarrow} : H_\bullet(M \times M;\, (LM \times LM, \mathrm{ev} \times \mathrm{ev})_+ \wedge_{M\times M} S_{M\times M}^{-\tau_{M\times M}}) \\ \to H_\bullet(M;\, \Delta^*(LM \times LM, \mathrm{ev} \times \mathrm{ev})_+ \wedge_M S_M^{-\tau_M}) \end{aligned}$$

which is just the umkehr map

$$LM^{-TM} \wedge LM^{-TM} \to (LM \times_M LM)^{-TM}$$

featuring in the Cohen–Jones construction [CJ02] of the loop product.

Chapter 5

Towards a field theory

In this chapter, we will prove our main result towards the construction of the field theories of Conjecture 1, namely the existence of a field-theory operation associated with a fixed cobordism W. This result is Theorem 41. While in that main result we only associate an operation to a single cobordism, it is natural to expect that, like in the case of the Freed–Hopkins–Teleman field-theory operations (see [FHT07a], Section 4), one should more generally be able to associate an operation to a family of cobordisms parametrized by a space. Indeed, much of our setup is tuned towards the construction of such an operation in the universal case where the family is the tautological one over the space of all cobordisms. The conjectural operation parametrized by this space is the composite map (5.56), and in Section 5.5 we will explain how this map would give rise to the Homological Conformal Field Theory operations we had in mind in Conjecture 1.

The chapter is structured as follows. In Section 5.1 we will define our cobordism categories, modeling them after the ones considered in [GMTW09]. Our field-theory operations depend on the choice of a universal R-orientation (see Definition 37), a notion that features Madsen–Tillmann spectra. In Section 5.2 we discuss Madsen–Tillmann spectra and relevant related spectra, and construct maps from the spaces of objects and morphisms in our cobordism categories to these spectra. It is through these maps that universal R-orientations interact with the cobordism category. The field-theory operation induced by a cobordism W arises from a pull–push construction in the diagram

$$\mathrm{map}(\partial_0 W, BG) \xleftarrow{s} \mathrm{map}(W, BG) \xrightarrow{t} \mathrm{map}(\partial_1 W, BG),$$

and in Section 5.3 we calculate the twisting involved in the pretransfer map induced by s by finding a fiber bundle approximation to the fibration s. In Section 5.4 we then construct our field-theory operations. Finally, in Section 5.5 we discuss conjectures that would get us closer to the construction of the field theories of Conjecture 1, and also comment on the relationship of our work to that of Freed, Hopkins and Teleman [FHT07a] and Chataur and Menichi [CM07].

5.1 Cobordism categories

In this section, we will describe briefly the cobordism categories $\mathcal{C}_d^+$ and $\mathcal{C}_d^+(K)$ relevant to the construction of our field theories, as well as their variants $\mathcal{C}_{d,\partial}^+$ and $\mathcal{C}_{d,\partial}^+(K)$. Intuitively, $\mathcal{C}_d^+$ is the category whose objects are oriented closed $(d-1)$-manifolds, and whose morphisms are the oriented cobordisms of such, and the category $\mathcal{C}_d^+(K)$ is similar, except that each object and morphism is equipped with a map to a space K. The variants $\mathcal{C}_{d,\partial}^+$ and $\mathcal{C}_{d,\partial}^+(K)$ are the subcategories of $\mathcal{C}_d^+$ and $\mathcal{C}_d^+(K)$, respectively, where every connected component of every cobordism has both an incoming and an outgoing boundary component.[1] Apart from the minor addition of the choice of a tubular neighborhood into the data making up an object or a morphisms, the cobordism categories $\mathcal{C}_d^+$ and $\mathcal{C}_d^+(K)$ are special cases of the cobordism categories considered in [GMTW09]. The genealogy of these cobordism categories goes back to at least [MT01].

As a first step toward defining our cobordism category of oriented d-manifolds in a background space K, consider the category $\widetilde{\mathcal{C}}_{d,L}^+(K)$ defined as follows. Let $\mathrm{Gr}_d(\mathbf{R}^L)$ and $\mathrm{Gr}_d^+(\mathbf{R}^L)$ denote the Grassmannians of unoriented and oriented d-dimensional linear subspaces of $\mathbf{R}^L$, respectively, and notice that an embedding of a d-dimensional

[1] Thus our $\mathcal{C}_{d,\partial}^+$ and $\mathcal{C}_{d,\partial}^+(K)$p are different from the positive boundary subcategories considered in [GMTW09], as in the positive boundary subcategories only the outgoing boundaries of the cobordisms are only required to be non-empty.

manifold M into $\mathbf{R}^L$ induces a canonical map

$$\tau_M : M \to \mathrm{Gr}_d(\mathbf{R}^L)$$

classifying the tangent bundle of M. Then the set of objects $\mathrm{Obj}\,\widetilde{\mathcal{C}}^+_{d,L}(K)$ consists of quadruples $(M, a, \tilde{\tau}_M, f)$, where $M \subset \mathbf{R}^L$ is a closed $(d-1)$-dimensional smooth submanifold of $\mathbf{R}^L$; a is a real number; $\tilde{\tau}_M$ is an orientation of M, by which we mean a lift of the canonical map

$$\tau_M : M \to \mathrm{Gr}_{d-1}(\mathbf{R}^L)$$

to a map

$$M \to \mathrm{Gr}^+_{d-1}(\mathbf{R}^L);$$

and f is a continuous map

$$f : M \to K.$$

The set of morphisms $\mathrm{Mor}\,\widetilde{\mathcal{C}}^+_{d,L}(K)$ is a disjoint union of a copy of $\mathrm{Obj}\,\widetilde{\mathcal{C}}^+_{d,L}(K)$ (giving the identity morphisms), and a set $\overline{\mathrm{Mor}}\,\widetilde{\mathcal{C}}^+_{d,L}(K)$ consisting of quintuples $(W, a_0, a_1, \tilde{\tau}_W, f)$, where a_0 and a_1 are real numbers; W is a compact d-dimensional submanifold of $[a_0, a_1] \times \mathbf{R}^L$ with boundary $\partial W = W \cap \{a_0, a_1\} \times \mathbf{R}^L$; $\tilde{\tau}_W$ is a lift of the canonical map

$$\tau_W : W \to \mathrm{Gr}_d(\mathbf{R} \times \mathbf{R}^L)$$

to $\mathrm{Gr}^+_d(\mathbf{R} \times \mathbf{R}^L)$; and f is a continuous map

$$f : W \to K.$$

We call $\partial_0 W = W \cap \{a_0\} \times \mathbf{R}^L$ the *incoming* and $\partial_1 W = W \cap \{a_1\} \times \mathbf{R}^L$ the *outgoing* boundary of W, and require that $W \cap [a_0, a_0 + \varepsilon[\times \mathbf{R}^L = [a_0, a_0 + \varepsilon[\times \partial_0 W$ and $W \cap]a_1 - \varepsilon, a_1] \times \mathbf{R}^L =]a_1 - \varepsilon, a_1] \times \partial_1 W$ for some small $\varepsilon > 0$. The source and target maps

$$s, t : \mathrm{Mor}\,\widetilde{\mathcal{C}}^+_{d,L}(K) \to \mathrm{Obj}\,\widetilde{\mathcal{C}}^+_{d,L}(K)$$

are given by restriction of the data to the incoming and outgoing boundaries, respectively, and composition is defined by concatenation. The sets $\mathrm{Mor}\,\widetilde{\mathcal{C}}^+_{d,L}(K)$ and

$\operatorname{Obj}\widetilde{\mathcal{C}}^+_{d,L}(K)$ can be given topologies in a natural way, making $\widetilde{\mathcal{C}}^+_{d,L}(K)$ a category internal to the category of topological spaces, but instead of reciting the long details here, we refer the reader to [GMTW09], with the advice that he or she should simply replace $\mathbf{R}^{d-1+\infty}$ in that discussion with $\mathbf{R}^L$ where appropriate.

The standard inclusion $\mathbf{R}^L \hookrightarrow \mathbf{R}^{L+1}$ induces a functor

$$\widetilde{\mathcal{C}}^+_{d,L}(K) \to \widetilde{\mathcal{C}}^+_{d,L+1}(K)$$

and in the limit as L goes to infinity, we obtain a category $\widetilde{\mathcal{C}}^+_d(K) = \widetilde{\mathcal{C}}^+_{d,\infty}(K)$ internal to topological spaces. Concretely, we can describe the category $\widetilde{\mathcal{C}}^+_d(K)$ simply by taking $L = \infty$ in the discussion above, where $\mathbf{R}^\infty$ is to be topologized as the colimit of the finite-dimensional spaces $\mathbf{R}^L$ for all L (so that every submanifold M or W of $\mathbf{R}^\infty$ is contained in some finite $\mathbf{R}^L$). This category $\widetilde{\mathcal{C}}^+_d(K)$ is a special case of the kind of cobordism categories considered in [GMTW09]. However, for our purposes it is desirable to augment the objects and morphisms in $\widetilde{\mathcal{C}}^+_d(K)$ with tubular neighborhoods. Thus, for finite L, we define a category $\mathcal{C}^+_{d,L}(K)$ as follows. The space of objects $\operatorname{Obj}\mathcal{C}^+_{d,L}(K)$ consists of quintuples $(M, a, \tilde{\tau}_M, f, \mathcal{O}_M)$, where $(M, a, \tilde{\tau}_M, f)$ is an object of $\widetilde{\mathcal{C}}^+_{d,L}(K)$ and $\mathcal{O}_M$ is a tubular neighborhood of the embedding $M \subset \mathbf{R}^L$, that is, an extension of the map $M \subset \mathbf{R}^L$ to a smooth open embedding of its normal bundle $\nu = \nu(M \subset \mathbf{R}^L)$ into $\mathbf{R}^L$ such that the composite map of vector bundles over the manifold M

$$\nu \to \tau_M \oplus \nu \approx \tau_\nu|_M \xrightarrow{d\mathcal{O}_M} \tau_{\mathbf{R}^L}|M \to \nu$$

is the identity. Unlike in the image one would typically draw of the situation, we do not require the image of $\mathcal{O}_M$ to be bounded. Similarly, the space $\overline{\operatorname{Mor}}\,\mathcal{C}^+_{d,L}(K)$ of non-identity morphisms consists of sextuples $(W, a_0, a_1, \tilde{\tau}_W, f, \mathcal{O}_W)$, where the first five components form a morphism in $\widetilde{\mathcal{C}}^+_{d,L}(K)$ and $\mathcal{O}_W$ is a tubular neighborhood of the embedding $W \subset [a_0, a_1] \times \mathbf{R}^L$. By our assumption on what W looks like near $\partial_0 W$, we know that for some $\varepsilon > 0$, the restriction of the normal bundle of $W \subset [a_0, a_1] \times \mathbf{R}^L$ to $W \cap [a_0, a_0 + \varepsilon[\times \mathbf{R}^L$ is canonically isomorphic to the product

$$[a_0, a_0 + \varepsilon[\times \nu(\partial_0 W \subset \mathbf{R}^L),$$

and we require that for t sufficiently close to a_0, the map $\mathcal{O}_W$ has the form

$$\mathcal{O}_W(t, v) = (t, \mathcal{O}_{\partial_0 W}(v))$$

for some tubular neighborhood $\mathcal{O}_{\partial_0 W}$ of $\partial_0 W \subset \mathbf{R}^L$; and similarly for $\partial_1 W$. Finally, we observe that the inclusion $\mathbf{R}^L \hookrightarrow \mathbf{R}^{L+1}$ induces a functor

$$\mathcal{C}^+_{d,L}(K) \to \mathcal{C}^+_{d,L+1}(K);$$

under this functor, a tubular neighborhood $\mathcal{O}_M : \nu(M \subset \mathbf{R}^L) \to \mathbf{R}^L$ is sent to the tubular neighborhood

$$\nu(M \subset \mathbf{R}^{L+1}) = \nu(M \subset \mathbf{R}^L) \times \mathbf{R} \xrightarrow{\mathcal{O}_M \times \mathrm{id}_{\mathbf{R}}} \mathbf{R}^L \times \mathbf{R} = \mathbf{R}^{L+1},$$

and similarly with tubular neighborhoods associated with morphisms. In the limit as L goes to infinity, we obtain a category $\mathcal{C}^+_d(K)$ internal to topological spaces.

Forgetting the tubular neighborhood gives functors

$$\mathcal{C}^+_{d,L}(K) \to \widetilde{\mathcal{C}}^+_{d,L}(K) \quad \text{and} \quad \mathcal{C}^+_d(K) \to \widetilde{\mathcal{C}}^+_d(K),$$

and as the choice of a tubular neighborhood is a contractible one, these functors induce homotopy equivalences between the spaces of objects and morphisms, and also between the classifying spaces of the categories. Given objects $c_i = (M_i, a_i, \tilde{\tau}_{M_i}, f_i, \mathcal{O}_{M_i})$, $i = 1, 2$, of $\mathcal{C}^+_d(K)$ with $a_0 < a_1$, by comparison with the corresponding space of morphisms in $\widetilde{\mathcal{C}}^+_d(K)$ and the discussion in [GMTW09], Section 5, we can identify the homotopy type of the space of maps from c_0 to c_1 as

$$\mathcal{C}^+_d(K)(c_0, c_1) \simeq \coprod_{[W]} E\mathrm{Diff}^+(W; \partial W) \times_{\mathrm{Diff}^+(W;\partial W)} \mathrm{map}^\partial(W, K) \tag{5.1}$$

where $\mathrm{map}^\partial(W, K)$ denotes the space of maps from W to K whose restrictions to $\partial_0 W = M_0$ and $\partial_1 W = M_1$ are given by f_0 and f_1, respectively, and the disjoint union is over all oriented cobordisms from M_0 to M_1, one in each oriented diffeomorphism class.

More often than the category $\mathcal{C}_d^+(K)$, we will actually need a variant $\mathcal{C}_{d,\partial}^+(K)$ defined in precisely the same way, except that both maps $\pi_0(\partial_0 W) \to \pi_0(W)$ and $\pi_0(\partial_1 W) \to \pi_0(W)$ are required to be surjections for the cobordisms W featured in the morphism. That is, we only consider those cobordisms W that have the property that all their connected components have at least one incoming and one outgoing boundary component. The analogue of equation (5.1) holds for $\mathcal{C}_{d,\partial}^+(K)$ when we simply restrict W to run through the representatives of oriented diffeomorphism types in this more restricted class of cobordisms.

The categories $\mathcal{C}_d^+$ and $\mathcal{C}_{d,\partial}^+$ can now be defined simply as

$$\mathcal{C}_d^+ = \mathcal{C}_d^+(\mathrm{pt}) \qquad \text{and} \qquad \mathcal{C}_{d,\partial}^+ = \mathcal{C}_{d,\partial}^+(\mathrm{pt}).$$

As there is only one map from any space to the one-point space pt, we can view $\mathcal{C}_d^+$ and $\mathcal{C}_{d,\partial}^+$ as the analogues of $\mathcal{C}_d^+(K)$ and $\mathcal{C}_{d,\partial}^+(K)$ obtained by simply omitting the map into K from the definitions, and this is how we will usually think about $\mathcal{C}_d^+$ and $\mathcal{C}_{d,\partial}^+$. As a special case of the homotopy equivalence (5.1), we have the equivalence

$$\mathcal{C}_d^+(c_0, c_1) \simeq \coprod_{[W]} B\mathrm{Diff}^+(W; \partial W) \tag{5.2}$$

where the disjoint union is over all diffeomorphism classes of oriented cobordisms from M_0 to M_1, and analogously for $\mathcal{C}_{d,\partial}^+(c_0, c_1)$.

Let $\mathcal{M} \to \mathrm{Obj}\,\mathcal{C}_d^+$ be the bundle whose fiber over the point $(M, a, \tilde{\tau}_M, \mathcal{O}_M)$ is the manifold M, and similarly define $\mathcal{W} \to \overline{\mathrm{Mor}}\,\mathcal{C}_d^+$ to be the bundle whose fiber over a point $(W, a_0, a_1, \tilde{\tau}_W, \mathcal{O}_W)$ is W. We denote by $\partial_0\mathcal{W}$ and $\partial_1\mathcal{W}$ the bundles over $\overline{\mathrm{Mor}}\,\mathcal{C}_d^+$ given by the pullbacks of $\mathcal{M}$ under the source and target maps

$$s, t : \overline{\mathrm{Mor}}\,\mathcal{C}_d^+ \to \mathrm{Obj}\,\mathcal{C}_d^+,$$

respectively. We then have obvious inclusions $\partial_0\mathcal{W} \hookrightarrow \mathcal{W}$ and $\partial_1\mathcal{W} \hookrightarrow \mathcal{W}$. We will use the same notations $\mathcal{M}$, $\mathcal{W}$, $\partial_0\mathcal{W}$ and $\partial_1\mathcal{W}$ for the corresponding bundles related to the category $\mathcal{C}^+_{d,\partial}$. Under the equivalence (5.2), the restriction of $\mathcal{W}$ over $\mathcal{C}^+_d(c_0, c_1)$ corresponds to the bundle

$$\coprod_{[W]} E\mathrm{Diff}^+(W;\partial W) \times_{\mathrm{Diff}^+(W;\partial W)} W \to \coprod_{[W]} B\mathrm{Diff}^+(W;\partial W)$$

and similarly for $\mathcal{C}^+_{d,\partial}$.

We will denote by

$$U : \mathcal{C}^+_d(K) \to \mathcal{C}^+_d.$$

the forgetful functor forgetting the map to K. The fiber of

$$U : \mathrm{Obj}\,\mathcal{C}^+_d(K) \to \mathrm{Obj}\,\mathcal{C}^+_d$$

over an object $(M, a, \tilde{\tau}_M, \mathcal{O}_M) \in \mathrm{Obj}\,\mathcal{C}^+_d$ is $\mathrm{map}(M, K)$, and similarly the fiber of

$$U : \mathrm{Mor}\,\mathcal{C}^+_d(K) \to \mathrm{Mor}\,\mathcal{C}^+_d$$

over a morphism $(W, a_0, a_1, \tilde{\tau}_W, \mathcal{O}_W) \in \mathrm{Mor}\,\mathcal{C}^+_d$ is $\mathrm{map}(W, K)$.

5.2 Madsen–Tillmann spectra

In this section, we will discuss the Madsen–Tillmann spectra relevant to the construction of our field theories, and will construct maps from the spaces of objects and morphisms of our cobordism categories into Madsen–Tillmann spectra and related spectra. Later on, these maps will provide the interface between universal R-orientations and the cobordism category. The principal spectra of interest are the Madsen–Tillmann spectrum $MTSO(d-1)$ (which we think of heuristically as an object knowledgeable about closed $(d-1)$-manifolds), the suspension spectrum

$\Sigma^\infty_+ BSO(d)$ (which we think of as an object which knows about d-manifolds with boundary), and a spectrum $Z(d)$ defined below (which we view as an object knowing about d-manifolds with boundary, where some of the boundary components have been labeled incoming and the rest as outgoing). Our intuitive perspective on the spectrum $MTSO(d-1)$ comes from an important theorem of Galatius, Madsen, Tillmann and Weiss [GMTW09], which identifies the homotopy type of the classifying space of the cobordism category $\mathcal{C}^+_{d-1}$ as $\Omega^{\infty-1} MTSO(d-1)$. We think of this result as saying that $\Omega^{\infty-1} MTSO(d-1)$ represents a kind of groupoid completion of the cobordism category, and from this point of view, the space $\Omega^\infty MTSO(d-1)$ should be thought of as the space of morphisms from the empty $(d-2)$-manifold to itself in the completed category. Thus we think of $\Omega^\infty MTSO(d-1)$ as related to the collection of closed $(d-1)$-manifolds (that is, the endomorphisms of the empty manifold in the cobordism category) by a completion process involving the ambient cobordism category. In particular, we would expect there to be a map that sends a closed $(d-1)$-manifold M to its image under the completion process, and it is in this way that we think of the adjoint of the map

$$\Sigma^\infty_+ \mathrm{Obj}\, \mathcal{C}^+_d \to MTSO(d-1)$$

constructed below. A generalization of the Galatius–Madsen–Tillmann–Weiss theorem to manifolds with boundary due to Genauer [Gen09] provides similar justification for our perspective on the spectrum $\Sigma^\infty_+ BSO(d)$. As for the spectrum $Z(d)$, to our knowledge the appropriate generalization of the Galatius–Madsen–Tillmann–Weiss theorem to manifolds with boundary components colored incoming or outgoing has not been proven, but we believe this to be because of lack of trying; in any case, we do have a natural map

$$\Sigma^\infty_+ \mathrm{Mor}\, \mathcal{C}^+_d \to Z(d)$$

as constructed below.

We now turn to the technical part of the section. The dimension d oriented Madsen–Tillmann spectrum $MTSO(d)$ is simply the Thom spectrum $BSO(d)^{-\gamma_d}$, where $\gamma_d \to BSO(d)$ is the universal oriented d-plane bundle. A prespectrum model for this spectrum is given by taking the L-th space of the spectrum to be

$$MTSO(d)_L = \mathrm{Gr}_d^+(\mathbf{R}^L)^{\gamma_d^\perp}$$

where $\gamma_d^\perp \to \mathrm{Gr}_d^+(\mathbf{R}^L)$ is the vector bundle whose fiber over a d-plane $V \in \mathrm{Gr}_d^+(\mathbf{R}^L)$ is the orthogonal complement of V inside $\mathbf{R}^L$. We observe that this model for $MTSO(d)$ arises from a spectrum over $BSO(d)$: Let $S_{BSO(d)}^{-\gamma_d}$ be the parametrized prespectrum with L-th space

$$(S_{BSO(d)}^{-\gamma_d})_L = i_! S_{\mathrm{Gr}_d^+(\mathbf{R}^L)}^{\gamma_d^\perp}$$

where $i : \mathrm{Gr}_d^+(\mathbf{R}^L) \hookrightarrow \mathrm{Gr}_d^+(\mathbf{R}^\infty) = BSO(d)$ is the map induced by the inclusion $\mathbf{R}^L \hookrightarrow \mathbf{R}^\infty$. Then, as suggested by the notation, we have

$$S_{BSO(d)}^{-\gamma_d} \wedge_{BSO(d)} S_{BSO(d)}^{\gamma_d} \simeq S_{BSO(d)},$$

and

$$MTSO(d) = r_! S_{BSO(d)}^{-\gamma_d}$$

where r is the map $r : BSO(d) \to \mathrm{pt}$.

The Madsen–Tillmann spectra $MTSO(d)$ and $MTSO(d-1)$ fit into a cofiber sequence

$$\Sigma^{-1}MTSO(d-1) \to MTSO(d) \to \Sigma_+^\infty BSO(d) \xrightarrow{\partial} MTSO(d-1) \tag{5.3}$$

where the first map is induced by the maps

$$\mathrm{Gr}_{d-1}^+(\mathbf{R}^L) \to \mathrm{Gr}_d^+(\mathbf{R} \times \mathbf{R}^L)$$

sending a $(d-1)$-plane V to the d-plane $\mathbf{R} \times V$. In more detail, this sequence can be derived as follows. As is well-known, the unit sphere bundle $S(\gamma_d)$ of $\gamma_d \to BSO(d)$ is homotopy equivalent to $BSO(d-1)$. Concretely, using the Grassmannian models for $BSO(d)$ and $BSO(d-1)$, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}^+_{d-1}(\mathbf{R}^\infty) & \xrightarrow{\simeq} & S(\gamma_d) \\ & \searrow \quad \swarrow{\scriptstyle p} & \\ & \mathrm{Gr}^+_d(\mathbf{R}\times\mathbf{R}^\infty) & \end{array}$$

where the map

$$\mathrm{Gr}^+_{d-1}(\mathbf{R}^\infty) \to \mathrm{Gr}^+_d(\mathbf{R}\times\mathbf{R}^\infty)$$

sends a $(d-1)$-plane V to the d-plane $\mathbf{R}\times V$, the map p is the projection, and the map

$$\mathrm{Gr}^+_{d-1}(\mathbf{R}^\infty) \xrightarrow{\simeq} S(\gamma_d)$$

sends a $(d-1)$-plane V to the point $(\mathbf{R}\times V,(1,0))$; an inverse homotopy equivalence to this map is given by the map sending a pair $(V,v)\in S(\gamma_d)$ consisting of an oriented d-plane $V\subset \mathbf{R}\times\mathbf{R}^\infty = \mathbf{R}^\infty$ and a unit vector $v\in V$ to the $(d-1)$-plane $V\cap\{v\}^\perp$ equipped with the induced orientation. Now the sphere bundle $S^{\gamma_d}_{BSO(d)}$ is homeomorphic to the fiberwise cofiber of the map

$$(S(\gamma_d),p)_+ \xrightarrow{p} (BSO(d),\mathrm{id})_+$$

of ex-spaces over $BSO(d)$, and the sequence (5.3) arises by applying the functor

$$r_!(- \wedge_{BSO(d)} S^{-\gamma_d}_{BSO(d)})$$

to the cofiber sequence

$$(S(\gamma_d),p)_+ \xrightarrow{p} (BSO(d),\mathrm{id})_+ \to S^{\gamma_d}_{BSO(d)} \to \Sigma_{BSO(d)}(S(\gamma_d),p)_+ \tag{5.4}$$

over $BSO(d)$ and making use of the homotopy equivalence $S(\gamma_d)\simeq BSO(d-1)$.

Given an object $(M,a,\tilde{\tau}_M,\mathcal{O}_M)\in \mathrm{Obj}\,\mathcal{C}^+_d$ coming from $\mathcal{C}^+_{d,L}$, we obtain a canonical map

$$S^L \to M^{\nu(M\subset \mathbf{R}^L)} \xrightarrow{\tilde{\tau}_M} \mathrm{Gr}^+_{d-1}(\mathbf{R}^L)^{\gamma^\perp_{d-1}}$$

where the first map is a Thom collapse map, and hence a point in

$$\Omega^L \mathrm{Gr}^+_{d-1}(\mathbf{R}^L)^{\gamma^\perp_{d-1}} \subset \Omega^\infty MTSO(d-1).$$

In this way, we obtain a map

$$\Sigma^\infty_+ \mathrm{Obj}\, \mathcal{C}^+_d \to MTSO(d-1). \tag{5.5}$$

Similarly, given a morphism $(W, a_0, a_1, \tilde{\tau}_W, \mathcal{O}_W) \in \mathrm{Mor}\, \mathcal{C}^+_d$ coming from $\mathcal{C}^+_{d,L}$, we obtain a map

$$[a_0, a_1]_+ \wedge S^L \to W^{\nu(W\subset [a_0,a_1]\times \mathbf{R}^L)} \xrightarrow{\tilde{\tau}_W} \mathrm{Gr}^+_d(\mathbf{R}\times \mathbf{R}^L)^{\gamma^\perp_d},$$

a construction that leads to a map

$$\Sigma^\infty_+ \mathrm{Mor}\, \mathcal{C}^+_d \to \Sigma MTSO(d)^{I_+} \tag{5.6}$$

making the diagram

$$\begin{array}{ccccc}
\Sigma^\infty_+ \mathrm{Obj}\, \mathcal{C}^+_d & \longrightarrow & MTSO(d-1) & \longrightarrow & \Sigma MTSO(d) \\
\uparrow{\scriptstyle s} & & & & \uparrow{\scriptstyle \mathrm{ev}_0} \\
\Sigma^\infty_+ \mathrm{Mor}\, \mathcal{C}^+_d & & \longrightarrow & & \Sigma MTSO(d)^{I_+} \\
\downarrow{\scriptstyle t} & & & & \downarrow{\scriptstyle \mathrm{ev}_1} \\
\Sigma^\infty_+ \mathrm{Obj}\, \mathcal{C}^+_d & \longrightarrow & MTSO(d-1) & \longrightarrow & \Sigma MTSO(d)
\end{array}$$

commute. The above diagram suggests that we could map $\Sigma^\infty_+ \mathrm{Mor}\, \mathcal{C}^+_d$ into a spectrum $Z(d)$ defined as the homotopy pullback of the diagram

$$MTSO(d-1) \to \Sigma MTSO(d) \leftarrow MTSO(d-1). \tag{5.7}$$

Indeed we shall do this, but rather than defining $Z(d)$ as the homotopy pullback of (5.7), it will be better for us to define $Z(d)$ in a more roundabout way that accords better with our understanding of the sequence (5.3) as a cofiber sequence arising from (5.4). Constructing $Z(d)$ and the map

$$\Sigma^\infty_+ \mathrm{Mor}\, \mathcal{C}_d^+ \to Z(d)$$

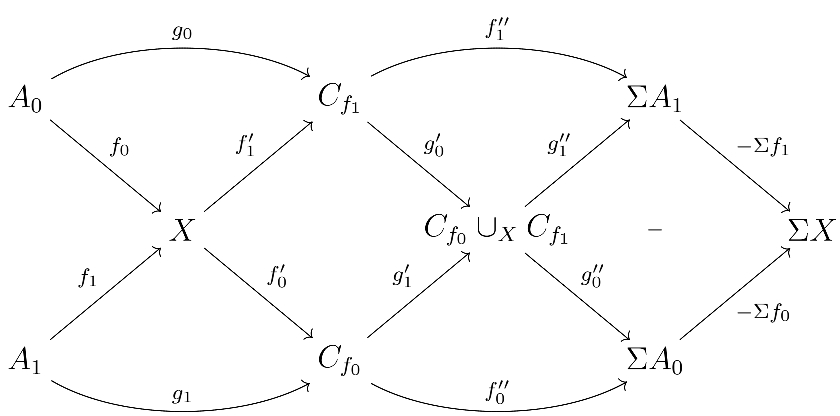
from this point of view will occupy a large part of the remainder of this section, but the effort will pay off for example in making it easy to check that diagram (5.28) commutes, a crucial component in the proofs of Lemmas 39 and 40 in Section 5.4.

Remark 31. The ideas behind the construction of the maps (5.5) and (5.6) go back to [MT01].

To begin the construction of $Z(d)$, recall that a map of $f : A \to X$ of pointed spaces generates a functorial (on the point-set level) cofiber sequence

$$A \xrightarrow{f} X \xrightarrow{f'} C_f \xrightarrow{f''} \Sigma A.$$

Given two maps $f_0 : A_0 \to X$ and $f_1 : A_1 \to X$, the cofiber sequences generated by f_0 and f_1 fit into a braid



where

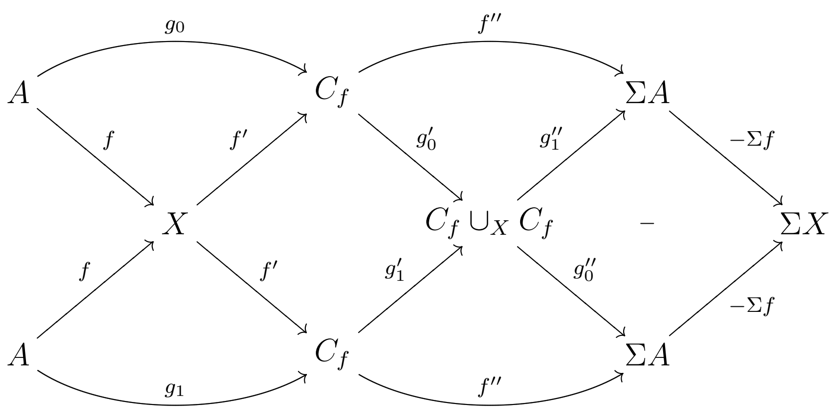
$$A_0 \xrightarrow{g_0} C_{f_1} \xrightarrow{g_0'} C_{f_0} \cup_X C_{f_1} \xrightarrow{g_0''} \Sigma A_0$$

and

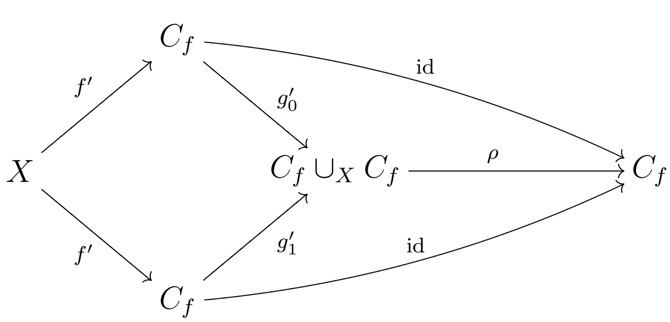
$$A_1 \xrightarrow{g_1} C_{f_0} \xrightarrow{g_1'} C_{f_0} \cup_X C_{f_1} \xrightarrow{g_1''} \Sigma A_1$$

are homeomorphic to the cofiber sequences generated by the maps $g_0 = f_1' f_0$ and $g_1 = f_0' f_1$, respectively. The square on the left is a pushout square, and hence homotopy cartesian, as the map f_0' is a cofibration. The diagram commutes strictly, except for the square on the right, which anticommutes up to a natural homotopy. Applying Σ^∞, we get a braid of distinguished triangles of spectra.

The case where f_0 and f_1 agree is of special interest. In that case, we obtain the diagram



Notice that $g_0 = g_1$ (and that both are null homotopic), while usually $g_0' \neq g_1'$ and $g_0'' \neq g_1''$; instead, in both cases the two maps are related by the self-homeomorphism of $C_f \cup_X C_f$ which switches the two copies of C_f. More explicitly, g_1'' is the map that collapses the first copy of C_f in $C_f \cup_X C_f$ to a point, and g_0'' is the map that collapses the second copy of C_f. The pushout property of the square on the left gives rise to a map $\rho : C_f \cup_X C_f \to C_f$ fitting into the diagram



so that ρ is a retraction of g_0' and g_1'. We can also define a map

$$\sigma : \Sigma A \to C_f \cup_X C_f \tag{5.8}$$

by setting σ equal to f on the equator, the canonical map to from CA to the second copy of C_f in $C_f \cup_X C_f$ above the equator, and the map from CA to the first copy of C_f below the equator. Then σ is a section of g_1'' and g_0'' in the sense that we have canonical homotopies

$$g_1''\sigma \simeq \mathrm{id}_{\Sigma A} \quad \text{and} \quad g_0''\sigma \simeq -\mathrm{id}_{\Sigma A}. \tag{5.9}$$

The following lemma relates the composite $f''\rho$ to g_0'' and g_1''.

Lemma 32. *In the above situation, the equation*

$$f''\rho = g_0'' + g_1''$$

holds after passing to the homotopy category of spectra.

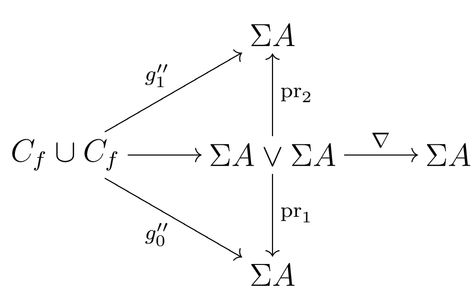
Proof. The composite

$$C_f \cup_X C_f \xrightarrow{\rho} C_f \xrightarrow{f''} \Sigma A$$

factors as

$$C_f \cup_X C_f \to (C_f \cup_X C_f)/X \approx \Sigma A \vee \Sigma A \xrightarrow{\nabla} \Sigma A$$

where ∇ is the fold map. The diagram



commutes, so the claim follows by passing to the homotopy category of spectra, where $\Sigma A \vee \Sigma A$ is a product and the map ∇ represents addition. □

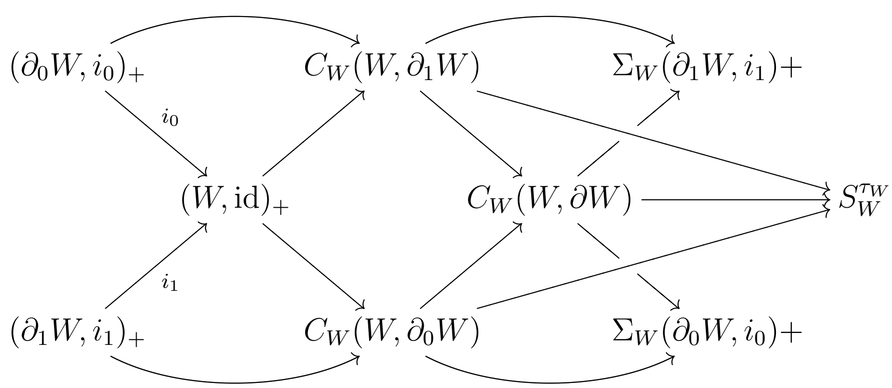
Suppose now $(W, a_0, a_1, \tilde{\tau}_W, \mathcal{O}_W)$ is a morphism in $\mathcal{C}_d^+$. Everything we have said about cofiber sequences so far works equally well in a parametrized setting when we form the cofibers and suspensions fiberwise. Given maps

$$A \overset{i}{\hookrightarrow} X \overset{p}{\to} B,$$

let us denote by $C_B(X, A)$ the fiberwise cofiber of the map

$$(A, pi)_+ \overset{i}{\to} (X, p)_+$$

of ex-spaces over B. Observe that the fiber of $C_B(X, A)$ over $b \in B$ is just the cone $C(X_b, A_b) = X_b \cup_{A_b} CA_b$; in particular, it is $(X_b)_+$ if A_b is empty. Let $i_k : \partial_k W \hookrightarrow W$, $k = 0.1$, be the inclusions. Then we have the following diagram of ex-spaces over W.



(5.10)

Here the braided part of the diagram is generated by the maps i_0 and i_1, and τ_W denotes the tangent bundle of W. The maps

$$C_W(W, \partial_k W) \to S_W^{\tau_W}$$

for $k = 0, 1$ are given by the zero section of $S_W^{\tau_W}$ (at the base of the cone), by the ∞-section of $S_W^{\tau_W}$ (at the top of the cone), and by the formula

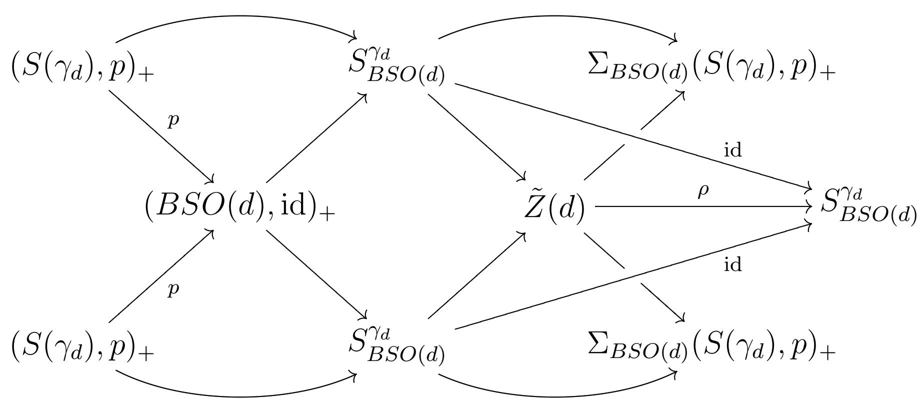
$$[x, t] \mapsto \left(x, \frac{t(1, 0)}{1 - t} \right)$$

for $x \in \partial_k W$ and cone coordinate $0 < t < 1$, where $(1, 0)$ is to be interpreted as a vector in $\mathbf{R} \times \mathbf{R}^\infty$; observe that by our assumption on what W looks like near its boundary, $(1, 0)$ is contained in τ_W and gives a unit normal vector for $\partial_k W$ at the point x. Finally, the map

$$C_W(W, \partial W) \to S_W^{\tau_W}$$

in the diagram is defined by the pushout property.

In addition to diagram (5.10), we have the following diagram of ex-spaces over $BSO(d)$.



(5.11)

Here the braided part of the diagram is generated by the map p, and ρ is defined by the pushout property. The counit map

$$(\tilde{\tau}_W)_! S_W^{\tau_W} = (\tilde{\tau}_W)_! (\tilde{\tau}_W)^* S^{\gamma_d}_{BSO(d)} \to S^{\gamma_d}_{BSO(d)}$$

and the maps generated by the horizontal composites in the diagram

$$\begin{array}{ccccc} \partial_k W & \xrightarrow{\tilde{\tau}_{\partial_k W}} & \mathrm{Gr}^+_{d-1}(\mathbf{R}^\infty) & \xrightarrow{\simeq} & S(\gamma_d) \\ \downarrow{\scriptstyle i_k} & & \downarrow & & \downarrow{\scriptstyle p} \\ W & \xrightarrow{\tilde{\tau}_W} & \mathrm{Gr}^+_d(\mathbf{R} \times \mathbf{R}^\infty) & = & BSO(d) \end{array}$$

together define a map from the $(\tilde{\tau}_W)_!$-pushforward of diagram (5.10) to the diagram (5.11). Applying the functor $r_!(- \wedge_{BSO(d)} S^{-\gamma_d}_{BSO(d)})$ to this map and making use of the homotopy equivalence $S(\gamma_d) \simeq BSO(d-1)$, we obtain a map from the diagram of spectra

$$\begin{tikzcd}
\Sigma^{-1}\partial_0 W^{-T\partial_0 W} \arrow[dr] \arrow[rr, bend left] & & (W^{-TW}, \Sigma^{-1}\partial_1 W^{-T\partial_1 W}) \arrow[dr] \arrow[rr, bend left] \arrow[drrr] & & \partial_1 W^{-T\partial_1 W} & \\
& W^{-TW} \arrow[ur] \arrow[dr] & & (W^{-TW}, \Sigma^{-1}\partial W^{-T\partial W}) \arrow[ur] \arrow[dr] \arrow[rr] & & \Sigma^\infty_+ W \\
\Sigma^{-1}\partial_1 W^{-T\partial_1 W} \arrow[ur] \arrow[rr, bend right] & & (W^{-TW}, \Sigma^{-1}\partial_0 W^{-T\partial_0 W}) \arrow[ur] \arrow[rr, bend right] \arrow[urrr] & & \partial_0 W^{-T\partial_0 W} &
\end{tikzcd} \tag{5.12}$$

to the diagram

$$\begin{tikzcd}
\Sigma^{-1}MTSO(d-1) \arrow[dr] \arrow[rr, bend left] & & \Sigma^\infty_+ BSO(d) \arrow[dr] \arrow[rr, bend left, "\partial"] \arrow[drrr, "\mathrm{id}"] & & MTSO(d-1) & \\
& MTSO(d) \arrow[ur] \arrow[dr] & & Z(d) \arrow[ur, "\tilde\partial_1"] \arrow[dr, "\tilde\partial_0"'] \arrow[rr, "\rho"] & & \Sigma^\infty_+ BSO(d) \\
\Sigma^{-1}MTSO(d-1) \arrow[ur] \arrow[rr, bend right] & & \Sigma^\infty_+ BSO(d) \arrow[ur] \arrow[rr, bend right, "\partial"'] \arrow[urrr, "\mathrm{id}"'] & & MTSO(d-1) &
\end{tikzcd} \tag{5.13}$$

where we are now working in the homotopy category of spectra. It is in this way that we define the spectrum $Z(d)$: we have

$$Z(d) = r_! \left(\tilde{Z}(d) \wedge_{BSO(d)} S^{-\gamma_d}_{BSO(d)} \right)$$

where $\tilde{Z}(d)$ is the spectrum over $BSO(d)$ defined as a pushout in diagram (5.11).

Remark 33. We note parenthetically that we can identify the homotopy type of $Z(d)$: together with the maps

$$\tilde{\partial}_0, \tilde{\partial}_1 : Z(d) \to MTSO(d-1),$$

the map

$$\rho : Z(d) \to \Sigma^\infty_+ BSO(d)$$

gives rise to two different splittings

$$(\rho, \tilde{\partial}_0) : Z(d) \xrightarrow{\simeq} \Sigma^\infty_+ BSO(d) \vee MTSO(d-1)$$

and

$$(\rho, \tilde{\partial}_1) : Z(d) \xrightarrow{\simeq} \Sigma^\infty_+ BSO(d) \vee MTSO(d-1).$$

Assuming that our morphism $(W, a_0, a_1, \tilde{\tau}_W, \mathcal{O}_W)$ comes from $\mathcal{C}^+_{d,L}$, we get a map

$$\begin{aligned} [a_0, a_1]_+ \wedge S^L &\to W^{\nu(W \subset [a_0,a_1] \times \mathbf{R}^L)} \\ &= r_! \left(S^0_W \wedge_W S_W^{\nu(W \subset [a_0,a_1] \times \mathbf{R}^L)} \right) \\ &\to r_! \left(C_W(W, \partial W) \wedge_W S_W^{\nu(W \subset [a_0,a_1] \times \mathbf{R}^L)} \right) \\ &= (W^{-TW}, \Sigma^{-1} \partial W^{-T\partial W})_{L+1} \end{aligned} \tag{5.14}$$

where the second map is induced by the inclusion $S^0_W \hookrightarrow C_W(W, \partial W)$ of ex-spaces over W. Using the cone coordinates in $C_W(W, \partial W)$, we can extend the map (5.14) to a map

$$S^{L+1} = (\mathbf{R} \cup \{\infty\}) \wedge S^L \to (W^{-TW}, \Sigma^{-1} \partial W^{-T\partial W})_{L+1} \tag{5.15}$$

which gives us a map of spectra

$$S \to (W^{-TW}, \Sigma^{-1} \partial W^{-T\partial W}), \tag{5.16}$$

a model for the Spanier-Whitehead dual of the map $W_+ \to S^0$. Composing with the map

$$(W^{-TW}, \Sigma^{-1} \partial W^{-T\partial W}) \to Z(d),$$

we obtain a map $S \to Z(d)$, or equivalently a point in $\Omega^\infty Z(d)$. In this way, we obtain a map

$$\Sigma^\infty_+ \overline{\mathrm{Mor}}\, \mathcal{C}^+_d \to Z(d). \tag{5.17}$$

Alternatively, we could observe that our constructions have been natural enough so that they can be performed fiberwise over $\overline{\mathrm{Mor}}\,\mathcal{C}_d^+$ to give a map

$$S_{\overline{\mathrm{Mor}}\,\mathcal{C}_d^+} \to \mathcal{DW} \to r^* Z(d) \tag{5.18}$$

where $\mathcal{DW}$ is the spectrum over $\overline{\mathrm{Mor}}\,\mathcal{C}_d^+$ whose fiber over the point $(W, a_0, a_1, \tilde{\tau}_W, \mathcal{O}_W)$ is the spectrum $(W^{-TW}, \Sigma^{-1}\partial W^{-T\partial W})$. Taking the adjoint of (5.18) now gives the map (5.17). Tracing through the constructions, the map σ of (5.8) gives rise to a map

$$MTSO(d-1) \to Z(d), \tag{5.19}$$

and we will use the composite

$$\Sigma_+^\infty \mathrm{Obj}\,\mathcal{C}_d^+ \to MTSO(d-1) \to Z(d)$$

to extend the map (5.17) to a map

$$\Sigma_+^\infty \mathrm{Mor}\,\mathcal{C}_d^+ = \Sigma_+^\infty \mathrm{Obj}\,\mathcal{C}_d^+ \vee \Sigma_+^\infty \overline{\mathrm{Mor}}\,\mathcal{C}_d^+ \to Z(d).$$

The maps $\tilde{\partial}_k$ are compatible with the source and target maps of $\mathcal{C}_d^+$ in the following sense.

Proposition 34. *In the diagram*

$$\begin{array}{ccccc} \Sigma_+^\infty \mathrm{Obj}\,\mathcal{C}_d^+ & \xleftarrow{s} & \Sigma_+^\infty \mathrm{Mor}\,\mathcal{C}_d^+ & \xrightarrow{t} & \Sigma_+^\infty \mathrm{Obj}\,\mathcal{C}_d^+ \\ \downarrow & - & \downarrow & & \downarrow \\ MTSO(d) & \xleftarrow{\tilde{\partial}_0} & Z(d) & \xrightarrow{\tilde{\partial}_1} & MTSO(d) \end{array} \tag{5.20}$$

the square on the right commutes up to homotopy while the square on the left anti-commutes up to homotopy.

Proof. On the summand $\Sigma_+^\infty \mathrm{Obj}\,\mathcal{C}_d^+$ of $\Sigma_+^\infty \mathrm{Mor}\,\mathcal{C}_d^+$, the claim follows from the homotopy equivalences (5.9). Let us concentrate on the summand $\Sigma_+^\infty \overline{\mathrm{Mor}}\,\mathcal{C}_d^+$. Suppose

$(W, a_0, a_1, \tilde{\tau}_W, \mathcal{O}_W) \in \overline{\mathrm{Mor}}\,\mathcal{C}_d^+$ comes from $\mathcal{C}_{d,L}^+$. We then have the commutative diagram

$$\begin{array}{ccccc}
S^{L+1} = (\mathbf{R}\cup\{\infty\})\wedge S^L & & & & \\
\Big\downarrow{\scriptstyle \pi} & & & & \\
r_!\left(C_W(W,\partial W)\wedge_W S_W^{\nu(W\subset[a_0,a_1]\times\mathbf{R}^L)}\right) & = & (W^{-TW},\Sigma^{-1}\partial W^{-T\partial W})_{L+1} & \longrightarrow & Z(d)_{L+1} \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle \tilde{\partial}_0} \\
r_!\left(\Sigma_W(\partial_0 W, i_0)_+ \wedge_W S_W^{\nu(W\subset[a_0,a_1]\times\mathbf{R}^L)}\right) & & & & \\
\Big\| & & & & \\
\Sigma\partial_0 W^{\nu(\partial_0 W\subset \mathbf{R}^L)} & = & (\partial_0 W^{-T\partial_0 W})_{L+1} & \longrightarrow & MTSO(d-1)_{L+1}
\end{array}$$

where π is the extension (5.15) of the map (5.14). The map π reverses the suspension coordinate $\mathbf{R}\cup\{\infty\}$ when mapping it to the cone coordinate in $C_W(W,\partial_0 W) \subset C_W(W,\partial W)$, and now inspection shows that the composite of the maps on the left in the above diagram is the negative of the suspension of the map

$$S^L \to \partial_0 W^{\nu(\partial_0 W\subset\mathbf{R}^L)}$$

associated with $s(W, a_0, a_1, \tilde{\tau}_W, \mathcal{O}_W) \in \mathrm{Obj}\,\mathcal{C}_d^+$. Thus the anticommutativity of the square on the left in diagram (5.20) follows. For $\partial_1 W$, the map π preserves the direction of the suspension coordinate when mapping it to the cone coordinate in $C_W(W,\partial_1 W)$, and the commutativity of the square on the right in diagram (5.20) follows from a diagram similar to the one above. □

Let us define

$$\partial_0 = -\tilde{\partial}_0 : Z(d) \to MTSO(d)$$

and

$$\partial_1 = \tilde{\partial}_1 : Z(d) \to MTSO(d).$$

With this notation, Proposition 34 states that the diagram

$$\begin{array}{ccccc} \Sigma^\infty_+ \mathrm{Obj}\,\mathcal{C}_d^+ & \xleftarrow{s} & \Sigma^\infty_+ \mathrm{Mor}\,\mathcal{C}_d^+ & \xrightarrow{t} & \Sigma^\infty_+ \mathrm{Obj}\,\mathcal{C}_d^+ \\ \downarrow & & \downarrow & & \downarrow \\ MTSO(d) & \xleftarrow{\partial_0} & Z(d) & \xrightarrow{\partial_1} & MTSO(d) \end{array} \tag{5.21}$$

commutes (in the homotopy category). Moreover, from the fiberwise version of Lemma 32 we obtain

$$\partial\rho \simeq \tilde{\partial}_0 + \tilde{\partial}_1 : Z(d) \to MTSO(d-1)$$

so that we get the equivalence

$$\partial\rho \simeq \partial_1 - \partial_0 : Z(d) \to MTSO(d-1). \tag{5.22}$$

So far we have succeeded in constructing maps

$$\Sigma^\infty_+ \mathrm{Obj}\,\mathcal{C}_d^+ \to MTSO(d-1) \quad \text{and} \quad \Sigma^\infty_+ \mathrm{Mor}\,\mathcal{C}_d^+ \to Z(d);$$

our next goal is to generalize these maps to maps

$$\Sigma^\infty_+ \mathrm{Obj}\,\mathcal{C}_d^+(K) \to MTSO(d-1) \wedge K_+ \quad \text{and} \quad \Sigma^\infty_+ \mathrm{Mor}\,\mathcal{C}_d^+(K) \to Z(d) \wedge K_+$$

involving a background space K. Let us first consider the map featuring $\mathrm{Obj}\,\mathcal{C}_d^+(K)$. Suppose $(M, a, \tilde{\tau}_M, \mathcal{O}_M) \in \mathcal{C}_d^+$ comes from $\mathcal{C}_{d,L}^+$. Then for a map $f : M \to K$, we have a map

$$S^L \to M^{\nu(M \subset \mathbf{R}^L)} \to \mathrm{Gr}_{d-1}^+(\mathbf{R}^L)^{\gamma_{d-1}^\perp} \wedge K_+$$

where the first map is the Pontryagin–Thom collapse map associated with the tubular neighborhood $\mathcal{O}_M$, and the second map is induced by the map

$$(\tilde{\tau}_M, f) : M \to \mathrm{Gr}_{d-1}^+(\mathbf{R}^L) \times K.$$

In this way, we get a map

$$\operatorname{map}(M,K)_+ \to \Omega^L(\mathrm{Gr}^+_{d-1}(\mathbf{R}^L)^{\gamma^\perp_{d-1}} \wedge K_+) \to \Omega^\infty(MTSO(d-1)\wedge K_+)$$

and more generally a map

$$\operatorname{Obj}\mathcal{C}^+_d(K)_+ \to \Omega^\infty(MTSO(d-1)\wedge K_+). \tag{5.23}$$

The desired map

$$\Sigma^\infty_+\operatorname{Obj}\mathcal{C}^+_d(K) \to MTSO(d-1)\wedge K_+.$$

is now the adjoint of (5.23).

Let us next consider the construction of the map

$$\Sigma^\infty_+\operatorname{Mor}\mathcal{C}^+_d(K) \to Z(d)\wedge K_+.$$

Suppose $(W, a_0, a_1, \tilde{\tau}_W, \mathcal{O}_W) \in \overline{\operatorname{Mor}}\,\mathcal{C}^+_d$ comes from $\mathcal{C}^+_{d,L}$. Given a map

$$\phi : (\tilde{\tau}_W)_! X \to Y$$

of ex-spaces over $BSO(d)$, we can construct a map

$$X \wedge \operatorname{map}(W,K)_+ \to (\tilde{\tau}_W)^*(Y\wedge K_+) \tag{5.24}$$

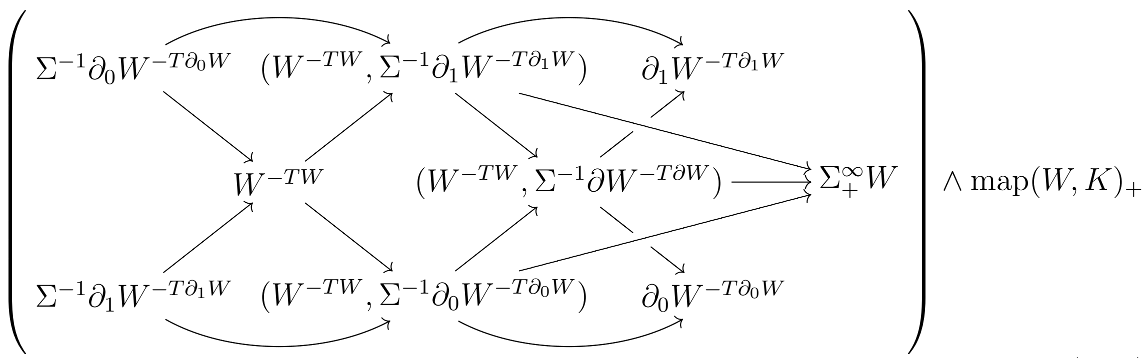
over W by sending a point $x\wedge f$ in the fiber over $w\in W$ to the point $\tilde{\phi}(x)\wedge f(w)$, where

$$\tilde{\phi} : X \to (\tilde{\tau}_W)^* Y$$

is the adjoint of ϕ. Taking the adjoint of the map (5.24), we see that ϕ induces a map

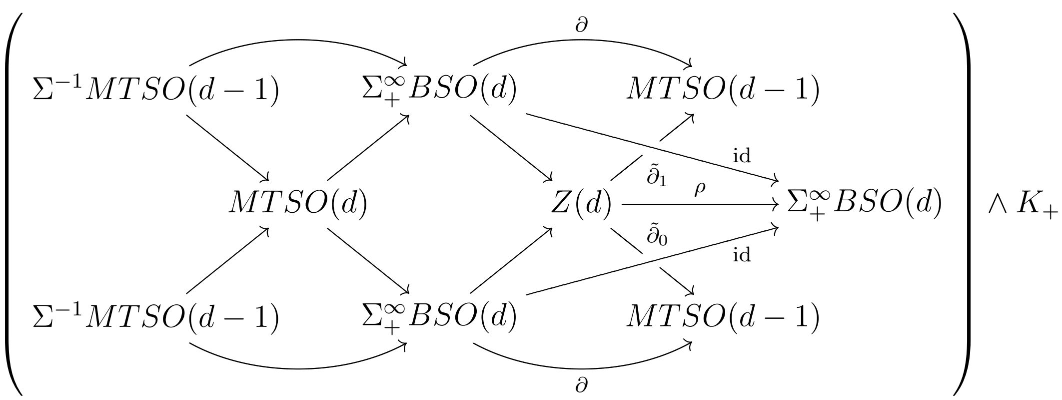
$$(\tilde{\tau}_W)_!(X\wedge \operatorname{map}(W,K)_+) \to Y\wedge K_+.$$

Applying this construction to the map from the $(\tilde{\tau}_W)_!$-pushforward of diagram (5.10) to the diagram (5.11), then applying the functor $r_!(-\wedge_{BSO(d)} S^{-\gamma_d}_{BSO(d)})$, and making use of the homotopy equivalence $S(\gamma_d) \simeq BSO(d-1)$, we get a map from the diagram of spectra



(5.25)

to the diagram



(5.26)

where again we are working in the homotopy category. In particular, we get a map

$$(W^{-TW}, \Sigma^{-1}\partial W^{-T\partial W}) \wedge \mathrm{map}(W, K)_+ \to Z(d) \wedge K_+,$$

and composing with the map

$$S \wedge \mathrm{map}(W, K)_+ \to (W^{-TW}, \Sigma^{-1}\partial W^{-T\partial W}) \wedge \mathrm{map}(W, K)_+$$

induced by the map (5.16), we get a map

$$\Sigma^\infty_+ \mathrm{map}(W, K) \to Z(d) \wedge K_+.$$

Our constructions have been sufficiently natural to work fiberwise over $\overline{\mathrm{Mor}}\,\mathcal{C}_d^+$, and so we get a map

$$\Sigma^\infty_{\overline{\mathrm{Mor}}\,\mathcal{C}_d^+}(\overline{\mathrm{Mor}}\,\mathcal{C}_d^+(K), U)_+ \to r^*(Z(d) \wedge K_+)+$$

whose adjoint gives us a map

$$\Sigma^\infty_+ \overline{\mathrm{Mor}}\,\mathcal{C}_d^+(K) \to Z(d) \wedge K_+.$$

Pairing this map with the map

$$\Sigma^\infty_+ \mathrm{Obj}\,\mathcal{C}_d^+(K) \to MTSO(d-1) \wedge K_+ \to Z(d) \wedge K_+$$

where the second map is induced by the map (5.19), we obtain the desired map

$$\Sigma^\infty_+ \mathrm{Mor}\,\mathcal{C}_d^+(K) = \Sigma^\infty_+ \mathrm{Obj}\,\mathcal{C}_d^+(K) \vee \Sigma^\infty_+ \overline{\mathrm{Mor}}\,\mathcal{C}_d^+(K) \to Z(d) \wedge K_+.$$

The analogue of Proposition 34 holds for the maps we have constructed, giving rise to the homotopy commutative diagram

$$\begin{array}{ccccc} \Sigma^\infty_+ \mathrm{Obj}\,\mathcal{C}_d^+(K) & \xleftarrow{s} & \Sigma^\infty_+ \mathrm{Mor}\,\mathcal{C}_d^+(K) & \xrightarrow{t} & \Sigma^\infty_+ \mathrm{Obj}\,\mathcal{C}_d^+(K) \\ \downarrow & & \downarrow & & \downarrow \\ MTSO(d) \wedge K_+ & \xleftarrow{\partial_0 \wedge K_+} & Z(d) & \xrightarrow{\partial_1 \wedge K_+} & MTSO(d) \wedge K_+ \end{array} \tag{5.27}$$

For later use, we also note that as part of the map from (5.25) to (5.26), we have the homotopy commutative square

$$\begin{array}{ccc} (W^{-TW}, \Sigma^{-1}\partial W^{-T\partial W}) \wedge \mathrm{map}(W, K) & \longrightarrow & Z(d) \wedge K_+ \\ \downarrow & & \downarrow \rho \\ \Sigma^\infty_+ (W \times \mathrm{map}(W, K)) & \xrightarrow{(\tilde{\tau}_W \mathrm{pr}_W, \mathrm{ev})} & \Sigma^\infty_+ (BSO(d) \times K) \end{array} \tag{5.28}$$

which is the fiber over $(W, a_0, a_1, \tilde{\tau}_W, \mathcal{O}_W)$ of the homotopy-commutative square

$$\begin{array}{ccc} \mathcal{DW} \wedge_{\overline{\mathrm{Mor}}\,\mathcal{C}_d^+} (\overline{\mathrm{Mor}}\,\mathcal{C}_d^+(K), U)_+ & \longrightarrow & r^*(Z \wedge K_+) \\ \downarrow & & \downarrow \\ \Sigma^\infty_{\overline{\mathrm{Mor}}\,\mathcal{C}_d^+}(\mathcal{W}, \pi_{\mathcal{W}})_+ \wedge_{\overline{\mathrm{Mor}}\,\mathcal{C}_d^+} (\overline{\mathrm{Mor}}\,\mathcal{C}_d^+(K), U)_+ & \longrightarrow & r^*\Sigma^\infty_+(BSO(d) \times K) \end{array} \tag{5.29}$$

of spectra over $\overline{\mathrm{Mor}}\,\mathcal{C}_d^+$. The commutativity of the square (5.28) will play an important role in the proofs of Lemmas 39 and 40 below.

5.3 A fiber bundle approximation

Let W be a connected cobordism between non-empty closed 1-manifolds. According to [CM07], Section 4.2, the fibers of the fibration

$$\mathrm{map}(W, BG) \xrightarrow{s} \mathrm{map}(\partial_0 W, BG). \tag{5.30}$$

given by restriction of maps to the incoming boundary have the homotopy type of $\Omega BG^{-\chi(W)} \simeq G^{-\chi(W)}$, whence Theorem 26 implies that the space $\mathrm{map}(W, BG)$ is Costenoble–Waner $\mathrm{map}(\partial_0 W, BG)$-dualizable. The aim of this section is to prove the following more precise result. In particular, it identifies the spectrum

$$\mathrm{ev}_{w_0}^* S_{BG}^{\chi(W)\mathrm{ad}(EG)}$$

over $\mathrm{map}(W, BG)$ as the twist involved in the pretransfer map induced by s. Here $\mathrm{ad}(EG)$ denotes the vector bundle

$$EG \times_G \mathfrak{g} \to BG$$

over BG, where $\mathfrak{g}$ is the Lie algebra of G equipped with the adjoint action,

$$\mathrm{ev}_{w_0} : \mathrm{map}(W, BG) \to BG$$

is the map given by evaluation against a fixed point w_0 of W, and $\chi(W)$ denotes the Euler characteristic of W.

Proposition 35. *Choose a basepoint $w_0 \in W$. Then the spectrum $\mathrm{ev}_{w_0}^* S_{BG}^{\chi(W)\mathrm{ad}(EG)}$ over $\mathrm{map}(W, BG)$ is a Costenoble–Waner $\mathrm{map}(\partial_0 W, BG)$-dual of $S_{\mathrm{map}(W,BG)}$.*

We will prove Proposition 35 by comparing the fibration (5.30) to a fiber bundle where the fiberwise Costenoble–Waner dual of the total space can be identified using the results in [MS06]. Let g be the genus of W, and let p and q denote the number of incoming and outgoing boundary components of W, respectively. Then we can view W as obtained from a regular $4g$-gon with $p+q$ open disks removed from the interior by identifying edges pairwise in the usual pattern. (If $g = 0$, we consider W as obtained from a disk with $p+q$ holes by collapsing the boundary circle to a point.) Choose a vertex of the polygon (or, if $g = 0$, pick any point on the boundary of the disk). Choose an embedding ϕ of the wedge sum $\bigvee^{p+q-1} S^1$ to the polygon (or disk) such that ϕ is smooth away from the basepoint $\bigvee^{p+q-1} S^1$, such that ϕ sends the basepoint of to the chosen vertex of the polygon (or to the chosen point of the boundary circle) and all other points to the interior of the polygon (or disk), and such that the image of ϕ restricted to each summand of $\bigvee^{p+q-1} S^1$ surrounds precisely one of the disks removed from the polygon (or disk), with all of the $p+q$ removed disks except for one corresponding to an outgoing boundary component surrounded by one of the summands of $\bigvee^{p+q-1} S^1$ in this way. Now combining ϕ with the embedding

$$\bigvee^{2g} S^1 \to W$$

given by the boundary edges of the polygon, we obtain an embedding

$$\bigvee^{n} S^1 = \Big(\bigvee^{p+q-1} S^1\Big) \vee \Big(\bigvee^{2g} S^1\Big) \to W, \tag{5.31}$$

where we have denoted $n = 2g + p + q - 1 = 1 - \chi(W)$. Expanding the disks removed from the polygon (or disk) gives a deformation retraction from W to the image of the embedding (5.31), and from the construction of (5.31), we obtain a homotopy-commutative diagram

$$\begin{array}{ccc} \coprod^p S^1 & \longrightarrow & \bigvee^n S^1 \\ \downarrow \approx & & \downarrow \simeq \\ \partial_0 W & \longrightarrow & W \end{array} \tag{5.32}$$

in which the top horizontal map factors as

$$\coprod^p S^1 \to \bigvee^p S^1 \hookrightarrow \bigvee^n S^1,$$

where the first map is given by the identification of the basepoints of the p copies of S^1 and the second map is given by inclusion of summands.

Diagram (5.32) induces a homotopy-commutative diagram

$$\begin{array}{ccc} \mathrm{map}(W, BG) & \xrightarrow{s} & \mathrm{map}(\partial_0 W, BG) \\ \downarrow \simeq & & \downarrow \approx \\ LBG^{\times_{BG} n} & \longrightarrow & LBG^p \end{array} \tag{5.33}$$

where $\times_{BG}$ indicates fiberwise product over BG, the map from LBG to BG being given by evaluation at the basepoint of S^1. It is well-known that

$$EG \times_G G^{\mathrm{ad}} \simeq LBG$$

and for example from [Gru07], Appendix A, we know that this equivalence can be realized by a zigzag of weak equivalences of spaces over BG where the structure map of each space over BG is a fibration. It follows that we have an equivalence

$$\mathrm{map}(S^1, BG)^{\times_{BG} n} \simeq (EG \times_G G^{\mathrm{ad}})^{\times_{BG} n} \approx EG \times_G (G^{\mathrm{ad}})^n$$

where the action of G on $(G^{\mathrm{ad}})^n$ is the diagonal one. Thus we get a homotopy-commutative diagram

$$\begin{array}{ccc} LBG^{\times_{BG} n} & \longrightarrow & LBG^p \\ \simeq \Big| & & \Big| \simeq \\ EG \times_G (G^{\mathrm{ad}})^n & \longrightarrow & (EG \times_G G^{\mathrm{ad}})^p \end{array} \tag{5.34}$$

where i-th coordinate function of the bottom horizontal map is induced by the projection

$$(G^{\mathrm{ad}})^n \to G^{\mathrm{ad}}$$

to the factor corresponding to the i-th incoming boundary component of W.

We would like to replace the bottom horizontal map in (5.34) by a fiber bundle. To this end, observe that we have the homotopy-commutative diagram

$$\begin{array}{ccc} EG \times_G (G^{\mathrm{ad}})^n & \xrightarrow[\simeq]{\tilde{\Delta}} & EG^p \times_G (G^{\mathrm{ad}})^n \\ \mathrm{pr} \Big\downarrow & & \\ EG \times_G (G^{\mathrm{ad}})^p & \swarrow_{\mathrm{pr}} & \\ \tilde{\Delta} \Big\downarrow \simeq & & \\ EG^p \times_G (G^{\mathrm{ad}})^p & & \\ \Big\downarrow & & \\ EG^p \times_{G^p} (G^{\mathrm{ad}})^p & & \\ \approx \Big| & & \\ (EG^p \times_G G^{\mathrm{ad}})^p & & \end{array} \tag{5.35}$$

where the maps labeled $\tilde{\Delta}$ are induced by the diagonal map

$$EG \xrightarrow{\simeq} EG^p$$

and the maps labeled pr are induced by the projection

$$(G^{\mathrm{ad}})^n \to (G^{\mathrm{ad}})^p$$

onto the factors corresponding to the incoming boundary components of W. The action of G on products is the diagonal one, and the second-last vertical map is induced by the diagonal inclusion $G \hookrightarrow G^p$. Notice that the composite of the vertical maps on the left is the bottom horizontal map of diagram (5.34)

We will show that composite map from the top right hand corner to the bottom in diagram (5.35) is a fiber bundle. Let

$$P = EG^p \times (G^{\mathrm{ad}})^p.$$

Then the coordinatewise right actions make P into a principal G^p-bundle. Denote

$$X = G^{n-p} \times (G^p/\Delta G)$$

where $\Delta G \subset G^p$ is the diagonal subgroup, and let G^p act on X from the left by

$$\bar{\gamma} \cdot (\bar{g}, \bar{g}'\Delta G) = (\gamma_1 \bar{g} \gamma_1^{-1}, \bar{\gamma}\bar{g}'\Delta G)$$

where $\bar{g} \in G^{n-p}$ and $\bar{\gamma} = (\gamma_1, \dots, \gamma_p)$ and $\bar{g}'$ are elements of G^p. Then the projection map

$$P \times_{G^p} X \to P/G^p \tag{5.36}$$

is homeomorphic to the composite

$$EG^p \times_G (G^{\mathrm{ad}})^n \to (EG \times_G G^{\mathrm{ad}})^p$$

in diagram (5.35). Thus we get the diagram

$$\begin{array}{ccc} EG \times_G (G^{\mathrm{ad}})^n & \longrightarrow & (EG \times_G G^{\mathrm{ad}})^p \\ \simeq\big\downarrow & & \big\downarrow\approx \\ P \times_{G^p} X & \longrightarrow & P/G^p \end{array} \tag{5.37}$$

Combining diagrams (5.33), (5.34) and (5.37), we get a homotopy-commutative diagram

$$\begin{array}{ccc} \mathrm{map}(W, BG) & \xrightarrow{s} & \mathrm{map}(\partial_0 W, BG) \\ \simeq \big| & & \big| \simeq \\ P \times_{G^p} X & \longrightarrow & P/G^p \end{array} \tag{5.38}$$

Thus we have succeeded in our goal of showing that s is equivalent to a fiber bundle.

Our next aim is to identify explicitly the Costenoble–Waner P/G^p-dual of $P \times_{G^p} X$. By Theorem 18.6.1, Proposition 18.3.2 and Corollary 19.4.4 of [MS06], this dual is $S^{-\tau_{\mathrm{vert}}}_{P \times_{G^p} X}$, where τ_{vert} is the vertical tangent bundle of (5.36), so our task is to identify the isomorphism type of τ_{vert}. We have a diffeomorphism

$$G^p/\Delta G \xrightarrow{\approx} G^{p-1}$$

sending $\bar{g}\Delta G$ with $\bar{g} = (g_1, \ldots, g_p)$ to the point $(g_2 g_1^{-1}, \ldots, g_p g_1^{-1})$, and under this diffeomorphism the left translation action of G^p corresponds to the action of G^p on G^{p-1} given by

$$(\gamma_1, \ldots, \gamma_p) \cdot (g_1, \ldots, g_{p-1}) = (\gamma_2 g_1 \gamma_1^{-1}, \ldots, \gamma_p g_{p-1} \gamma_1^{-1}).$$

Crossing with the identity map of G^{n-p}, the diffeomorphism gives us a diffeomorphism

$$X = G^{n-p} \times (G^p/\Delta G) \approx G^{n-1}$$

Let $V = \mathfrak{g}^{n-1}$, and let G^p act on V via the projection

$$G^p \to G$$

onto the first factor and the diagonal action of G on $\mathfrak{g}^{n-1}$. By inspection, we have a pullback square of G^p-spaces

$$\begin{array}{ccc} TX & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & \mathrm{pt} \end{array} \tag{5.39}$$

and hence, by applying $P \times_{G^p} (-)$, a pullback square

$$\begin{array}{ccc} \tau_{\text{vert}} & \longrightarrow & P \times_{G^p} V \\ \downarrow & & \downarrow \\ P \times_{G^p} X & \longrightarrow & P/G^p \end{array} \tag{5.40}$$

In particular, the vertical tangent bundle τ_{vert} is a pullback of a bundle over its base space.

To analyze $P \times_{G^p} V$, observe that

$$P \times_{G^p} V = (EG^p \times (G^{\text{ad}})^p) \times_{G^p} V = EG^p \times_{G^p} ((G^{\text{ad}})^p \times V)$$

where we have converted the right action of G^p on $(G^{\text{ad}})^p$ to a left one. We have the pullback square of G^p-spaces

$$\begin{array}{ccc} (G^{\text{ad}})^p \times V & \longrightarrow & V \\ \downarrow & & \downarrow \\ (G^{\text{ad}})^p & \longrightarrow & \text{pt} \end{array} \tag{5.41}$$

and applying $EG^p \times_{G^p} (-)$, we obtain a pullback square

$$\begin{array}{ccc} P \times_{G^p} V & \longrightarrow & EG^p \times_{G^p} V \\ \downarrow & & \downarrow \\ P/G^p & \longrightarrow & BG^p \end{array} \tag{5.42}$$

Furthermore, recalling the action of G^p on V, we see that there is a pullback square

$$\begin{array}{ccc} EG^p \times_{G^p} V & \longrightarrow & (n-1)\text{ad}(EG) \\ \downarrow & & \downarrow \\ BG^p & \xrightarrow{\text{pr}_1} & BG \end{array} \tag{5.43}$$

where pr_1 is projection onto the first coordinate. Combining the pullback squares

(5.40), (5.42) and (5.43), we obtain the pullback square

$$\begin{CD} \tau_{\mathrm{vert}} @>>> (n-1)\mathrm{ad}(EG) \\ @VVV @VVV \\ P \times_{G^p} X @>>> BG \end{CD} \tag{5.44}$$

Under the equivalence

$$\mathrm{map}(W, BG) \simeq P \times_{G^p} X$$

of (5.38), the classifying map

$$P \times_{G^p} X \to BG$$

obtained in (5.44) for τ_{vert} corresponds to the map

$$\mathrm{ev}_w : \mathrm{map}(W, BG) \to BG \tag{5.45}$$

given by evaluation against the point w that is the image of the basepoint of $\bigvee^n S^1$ under the embedding (5.31). Since W is path connected, the map ev_w and the map ev_{w_0} of Proposition 35 are homotopic. Recalling that $\chi(W) = 1 - n$, the claim in Proposition 35 follows.

Remark 36. In the above argument, the first coordinate of G^p seemed to play a special role, but this is of course illusory. We have a commutative diagram

$$\begin{CD} @. P \times_{G^p} X \\ @. @VV{\approx}V \\ EG \times_G (G^{\mathrm{ad}})^n @>{\tilde{\Delta}}>{\simeq}> EG^p \times_G (G^{\mathrm{ad}})^n \\ @VVV @VVV \\ BG @>{\Delta}>> BG^p \\ @. @VV{\mathrm{pr}_i}V \\ @. BG \end{CD}$$

where the map pr_i is projection onto the i-th coordinate. For $i = 1$, the composite of the right hand vertical maps is the classifying map for τ_{vert} obtained in (5.44). However, it turns out that the composites of the right hand vertical maps are homotopic for all i: one just needs to observe that the composite map from $EG \times_G (G^{\mathrm{ad}})^n$ to BG in the diagram is independent of i.

5.4 The field-theory operations

In this section, we will finally construct our field-theory operations. Our main result is Theorem 41, which, given a commutative S-algebra R and a piece of orientation data we call a universal R-orientation, asserts the existence of a field theory operation associated to a single cobordism W. This operation goes from a twisted version of the R-homology of $\mathrm{map}(\partial_0 W, BG)$ to a twisted version of the R-homology of $\mathrm{map}(\partial_1 W, BG)$, and arises from a pull–push construction in the diagram

$$\mathrm{map}(\partial_0 W, BG) \xleftarrow{\ s\ } \mathrm{map}(W, BG) \xrightarrow{\ t\ } \mathrm{map}(\partial_1 W, BG),$$

where the maps s and t are given by restriction. The role of the universal R-orientation is to give rise to the R-theory twistings of $\mathrm{map}(\partial_0 W, BG)$ and $\mathrm{map}(\partial_1 W, BG)$, and to provide a compatibility relation between the pullbacks of these twistings to the space $\mathrm{map}(W, BG)$.

To prepare the way to the definition of the notion of a universal R-orientation, let us begin by introducing notation. Let $\sigma_{\mathfrak{g}}$ denote the composite

$$\sigma_{\mathfrak{g}} : \Sigma^\infty_+(BSO(2) \times BG) \xrightarrow{\mathrm{pr}} \Sigma^\infty_+ BG \xrightarrow{\mathfrak{g}} ko$$

where the latter map classifies the vector bundle $\mathrm{ad}(EG) = (EG \times_G \mathfrak{g} \to BG)$, where $\mathfrak{g}$ denotes the Lie algebra of G. Likewise, let σ_d be the composite

$$\sigma_d : \Sigma^\infty_+(BSO(2) \times BG) \to \Sigma^\infty_+ \mathrm{pt} \xrightarrow{d} ko$$

where d is the dimension of G and the latter map classifies the d-dimensional trivial bundle. We have a short exact sequence of KO-groups

$$0 \to \widetilde{KO}(BSO(2)_+ \wedge BG) \to KO(BSO(2) \times BG) \to KO(BSO(2)) \to 0,$$

and we let

$$\bar{\sigma}_{\mathfrak{g}} : \Sigma^\infty_+ BSO(2) \wedge BG \to ko$$

be a map representing a lift of the difference $\sigma_{\mathfrak{g}} - \sigma_d$ from the group in the middle to the one on the left. Let R be an commutative S-algebra, that is, a commutative monoid in some suitable category of spectra. Inspired by Freed, Hopkins and Teleman [FHT07a], we make the following definition.

Definition 37. A *universal R-orientation* is a null homotopy of the composite

$$MTSO(2) \wedge BG \to \Sigma^\infty_+ BSO(2) \wedge BG \xrightarrow{\bar{\sigma}_{\mathfrak{g}}} ko \to \mathrm{line}_\bullet(R)$$

Two universal R-orientations are called equivalent if the null homotopies are homotopy equivalent.

Here the spectrum $\mathrm{line}_\bullet(R)$ and the map $ko \to \mathrm{line}_\bullet(R)$ were discussed in Section 3.6. We observe that by standard arguments, if a universal R-orientation exists, then the set of equivalence classes of universal R-orientations is a torsor over the abelian group

$$[\Sigma MTSO(2) \wedge BG, \mathrm{line}_\bullet(R)],$$

and that a universal R-orientation is equivalent data to the choice of a map ε together with a homotopy making the diagram

$$\begin{array}{ccc} \Sigma^\infty_+ BSO(2) \wedge BG & \longrightarrow & MTSO(1) \wedge BG \\ {\scriptstyle \bar{\sigma}_{\mathfrak{g}}}\downarrow & \nearrow & \downarrow{\scriptstyle \varepsilon} \\ ko & \longrightarrow & \mathrm{line}_\bullet(R) \end{array} \tag{5.46}$$

commutative. In fact, it is in terms of the dotted data in diagram (5.46) that we will usually think of a universal R-orientation.

The notion of a universal R-orientation would not be a very useful one if such orientations did not exist. Luckily, the existence of universal R-orientations is fairly common, as demonstrated by the following proposition. Recall that a cohomology theory is called *complex oriented* if every complex vector bundle comes equipped with a canonical orientation with respect to the theory. Examples of such theories include ordinary cohomology $H\mathbf{Z}$, complex K-theory K, and complex cobordism MU.

Proposition 38. *Suppose R represents a complex-oriented cohomology theory. Then there exists a canonical universal R-orientation.*

Proof. The proof of the first part of Theorem 3.24 in [FHT07a] goes through with $\mathrm{line}_\bullet(R)$ in place of the spectrum $\mathrm{pic}_g K$ considered in [FHT07a], giving the desired result. □

Let us now fix a universal R-orientation. The composite

$$\Sigma^\infty_+ \mathrm{Obj}\, \mathcal{C}^+_{2,\partial}(BG) \to \Sigma^\infty_+ \mathrm{Obj}\, \mathcal{C}^+_2(BG) \to MTSO(1) \wedge BG_+ \to \\ \to MTSO(1) \wedge BG \xrightarrow{\varepsilon} \mathrm{line}_\bullet(R)$$

defines an R-line bundle $\mathcal{E}$ over the space $\mathrm{Obj}\, \mathcal{C}^+_{2,\partial}(BG)$. Likewise, the composites

$$\Sigma^\infty_+ \mathrm{Mor}\, \mathcal{C}^+_{2,\partial}(BG) \to \Sigma^\infty_+ \mathrm{Mor}\, \mathcal{C}^+_2(BG) \to Z(2) \wedge BG_+ \xrightarrow{\rho \wedge BG_+} \\ \xrightarrow{\rho \wedge BG_+} \Sigma^\infty_+ BSO(2) \wedge BG_+ = \Sigma^\infty_+ (BSO(2) \times BG) \xrightarrow{\sigma_{\mathfrak{g}}} ko$$

and

$$\Sigma^\infty_+ \mathrm{Mor}\, \mathcal{C}^+_{2,\partial}(BG) \to \Sigma^\infty_+ \mathrm{Mor}\, \mathcal{C}^+_2(BG) \to Z(2) \wedge BG_+ \xrightarrow{\rho \wedge BG_+} \\ \xrightarrow{\rho \wedge BG_+} \Sigma^\infty_+ BSO(2) \wedge BG_+ = \Sigma^\infty_+ (BSO(2) \times BG) \xrightarrow{\sigma_d} ko$$

give rise to S-line bundles $\mathcal{T}_{\mathfrak{g}}$ and $\mathcal{T}_d$ over the space $\mathrm{Mor}\, \mathcal{C}^+_{2,\partial}(BG)$, respectively. We observe that $\mathcal{T}_d$ is the pullback of the S-line bundle $\mathcal{T}'_d$ over $\mathrm{Mor}\, \mathcal{C}^+_{2,\partial}$ classified by the composite

$$\Sigma^\infty_+ \mathrm{Mor}\, \mathcal{C}^+_{2,\partial} \longrightarrow \Sigma^\infty_+ \mathrm{Mor}\, \mathcal{C}^+_2 \longrightarrow Z(2) \xrightarrow{\rho} \Sigma^\infty_+ BSO(2) \longrightarrow \Sigma^\infty_+ \mathrm{pt} \xrightarrow{d} ko.$$

Frequently, we will need the just the parts of $\mathcal{E}$, $\mathcal{T}_{\mathfrak{g}}$ and $\mathcal{T}_d$ that lie over a fixed object or morphism of $\mathcal{C}^+_{2,\partial}$. Given an object $M = (M, a, \tilde{\tau}_M, \mathcal{O}_M)$ of $\mathcal{C}^+_{2,\partial}$, we will denote by $\mathcal{E}_M$ the restriction of $\mathcal{E}$ to the fiber $\mathrm{map}(M, BG)$ over $M \in \mathrm{Obj}\, \mathcal{C}^+_{2,\partial}$ of the forgetful map

$$U : \mathrm{Obj}\, \mathcal{C}^+_{2,\partial}(BG) \to \mathrm{Obj}\, \mathcal{C}^+_{2,\partial}.$$

Similarly, given a morphism $W = (W, a_0, a_1, \tilde{\tau}_W, \mathcal{O}_W)$ of $\mathcal{C}^+_{2,\partial}$, we define $\mathcal{T}_{\mathfrak{g},W}$ and $\mathcal{T}_{d,W}$ to be the restrictions of $\mathcal{T}_{\mathfrak{g}}$ and $\mathcal{T}_d$ to the fiber $\mathrm{map}(W, BG)$ over the point $W \in \mathrm{Mor}\, \mathcal{C}^+_{2,\partial}$ of the forgetful map

$$U : \mathrm{Mor}\, \mathcal{C}^+_{2,\partial}(BG) \to \mathrm{Mor}\, \mathcal{C}^+_{2,\partial}.$$

Furthermore, we will denote by $\mathcal{T}'_{d,W}$ the fiber of $\mathcal{T}'_d$ over the point $W \in \mathrm{Mor}\, \mathcal{C}^+_{2,\partial}$. We observe that $\mathcal{T}_{d,W}$ is then a pullback of $\mathcal{T}'_{d,W}$, and hence a trivial S-line bundle.

Consider the diagram

$$\begin{array}{ccc}
\Sigma^\infty_+ \mathrm{Mor}\, \mathcal{C}^+_{2,\partial}(BG) & \xrightarrow{(s,t)} & (\Sigma^\infty_+ \mathrm{Obj}\, \mathcal{C}^+_{2,\partial}(BG))^{\times 2} \\
\downarrow & & \downarrow \\
\Sigma^\infty_+ \mathrm{Mor}\, \mathcal{C}^+_2(BG) & \xrightarrow{(s,t)} & (\Sigma^\infty_+ \mathrm{Obj}\, \mathcal{C}^+_2(BG))^{\times 2} \\
\downarrow & & \downarrow \\
Z(2) \wedge BG_+ & \xrightarrow{(\partial_0, \partial_1)} & (MTSO(1) \wedge BG_+)^{\times 2} \\
\rho \downarrow & & \downarrow (-\mathrm{id}, \mathrm{id}) \\
\Sigma^\infty_+ BSO(2) \wedge BG_+ & \xrightarrow{\partial} & MTSO(1) \wedge BG_+ \\
\downarrow & & \downarrow \\
\Sigma^\infty_+ BSO(2) \wedge BG & \xrightarrow{\partial} & MTSO(1) \wedge BG \\
\bar{\sigma}_{\mathfrak{g}} \downarrow & \nearrow & \downarrow \varepsilon \\
ko & \longrightarrow & \mathrm{line}_\bullet(R)
\end{array}$$

Each square in the diagram is at least homotopy commutative. The second square from the top commutes by diagram (5.27), and for the third square from top, the commutativity follows from equation (5.22). For the bottom square, the map ε and the indicated homotopy are given by the universal R-orientation. For the remaining squares, the commutativity is obvious. From the homotopy commutativity of the diagram, we deduce the existence of an equivalence

$$(\mathcal{T}_{\mathfrak{g}} \wedge_{\operatorname{Mor} \mathcal{C}^{+}_{2,\partial}(BG)} \mathcal{T}_d^{-1}) \wedge R \simeq t^*\mathcal{E} \wedge_{R, \operatorname{Mor} \mathcal{C}^{+}_{2,\partial}(BG)} (s^*\mathcal{E})^{-1} \tag{5.47}$$

of R-line bundles over $\operatorname{Mor} \mathcal{C}^{+}_{2,\partial}(BG)$, and we can rewrite this equivalence as an equivalence

$$s^*\mathcal{E} \wedge_{\operatorname{Mor} \mathcal{C}^{+}_{2,\partial}(BG)} \mathcal{T}_{\mathfrak{g}} \simeq t^*\mathcal{E}. \wedge_{\operatorname{Mor} \mathcal{C}^{+}_{2,\partial}(BG)} \mathcal{T}_d \tag{5.48}$$

Let us now fix a non-identity morphism $W = (W, a_0, a_1, \tilde{\tau}_W, \mathcal{O}_W)$ of $\mathcal{C}^{+}_{2,\partial}$. Then the source and target maps in the category $\mathcal{C}^{+}_{2,\partial}(BG)$ restrict to give a diagram

$$\operatorname{map}(\partial_0 W, BG) \xleftarrow{\ s\ } \operatorname{map}(W, BG) \xrightarrow{\ t\ } \operatorname{map}(\partial_1 W, BG), \tag{5.49}$$

and the equivalence (5.48) restricts to an equivalence

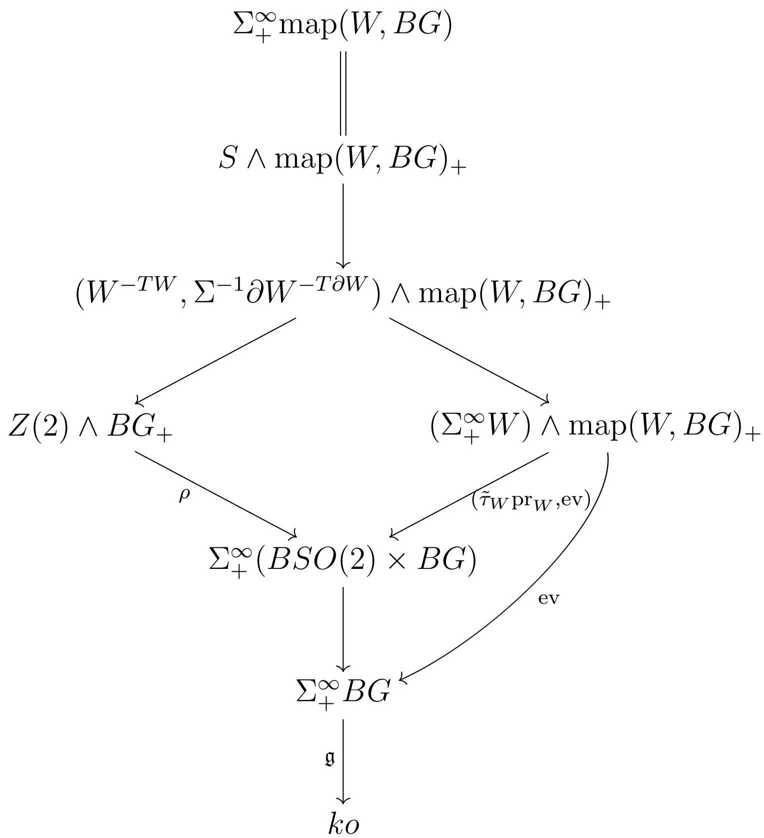
$$s^*\mathcal{E}_{\partial_0 W} \wedge_{\operatorname{map}(W,BG)} \mathcal{T}_{\mathfrak{g},W} \simeq t^*\mathcal{E}_{\partial_1 W} \wedge_{\operatorname{map}(W,BG)} \mathcal{T}_{d,W} \tag{5.50}$$

of R-line bundles over $\operatorname{map}(W, BG)$.

The following lemma is a key step in the construction of the field theory operations, as it identifies $\mathcal{T}_{\mathfrak{g},W}$ as the twisting associated with the umkehr map induced by the map s in diagram (5.49).

Lemma 39. *The spectrum $\mathcal{T}_{\mathfrak{g},W}$ over the mapping space $\operatorname{map}(W, BG)$ is a Costenoble–Waner $\operatorname{map}(\partial_0 W, BG)$-dual of $S_{\operatorname{map}(W,BG)}$.*

Proof. By diagram (5.28), the diamond in the diagram



(5.51)

commutes up to homotopy. Observe that the spectrum $\mathcal{T}_{\mathfrak{g},W}$ over $\mathrm{map}(W, BG)$ is classified by the composite map down from the top to the bottom of the diagram along the left-hand route. Let us first consider the case where W is connected. From Spanier–Whitehead duality theory, we know that the map

$$S \to (W^{-TW}, \Sigma^{-1}\partial W^{-T\partial W}) \to \Sigma^\infty_+ W$$

that features in the composite down the right-hand route represents $\chi(W)$ times the generator of $\pi_0^s(W_+) \approx \mathbf{Z}$. On the other hand, the generator of $\pi_0^s(W_+)$ is represented by any map of spaces

$$w_0 : S^0 \to W_+$$

where the non-basepoint point of S^0 is sent to a point w_0 of W. Thus the composite down along the right-hand route is equivalent to $\chi(W)$ times the map

$$\Sigma^\infty S^0 \wedge \mathrm{map}(W, BG)_+ \xrightarrow{w_0} \Sigma^\infty_+(W \times \mathrm{map}(W, BG)) \xrightarrow{\mathrm{ev}} \Sigma^\infty_+ BG \xrightarrow{\mathfrak{g}} ko$$

whence the claim follows from Proposition 35. This proves the lemma in the case where W is connected.

Suppose now W consists of connected components $W_1, \ldots, W_k$. Then our fibration

$$\mathrm{map}(W, BG) \to \mathrm{map}(\partial_0 W, BG)$$

is the product of the fibrations

$$\mathrm{map}(W_i, BG) \to \mathrm{map}(\partial_0 W_i, BG),$$

$i = 1, \ldots, k$. Let

$$\pi_i : \mathrm{map}(W, BG) \approx \prod_{i=1}^{k} \mathrm{map}(W_i, BG) \to \mathrm{map}(W_i, BG)$$

be the projection. In general, if $E_1 \to B_1$ and $E_2 \to B_2$ are fibrations whose fibers are Costenoble–Waner dualizable, and T_i is a Costenoble–Waner B_i-dual for S_{E_i}, $i = 1, 2$, then the product map

$$E_1 \times E_2 \to B_1 \times B_2$$

is a fibration with Costenoble–Waner dualizable fibers, and $S_{E_1 \times E_2}$ has $B_1 \times B_2$-dual

$$\bar{\pi}_1^* T_1 \wedge_{E_1 \times E_2} \bar{\pi}_2^* T_2$$

where

$$\bar{\pi}_i : E_1 \times E_2 \to E_i,$$

$i = 1, 2$ are the projections. In terms of classifying maps, we thus want to show that $\mathcal{T}_{\mathfrak{g},W}$ is classified by the sum

$$\sum_{i=1}^{k} \pi_i^* \mathcal{T}^{\mathrm{cl}}_{\mathfrak{g},W_i}$$

where $\mathcal{T}^{\mathrm{cl}}_{\mathfrak{g},W_i}$ is a classifying map for $\mathcal{T}_{\mathfrak{g},W_i}$.

In the commutative diagram

$$\begin{array}{ccc}
\Sigma^\infty_+ \mathrm{map}(W, BG) & \xrightarrow{\pi_i} & \Sigma^\infty_+ \mathrm{map}(W_i, BG) \\
\| & & \| \\
S \wedge \mathrm{map}(W, BG)_+ & \xrightarrow{\pi_i} & S \wedge \mathrm{map}(W_i, BG)_+ \\
\downarrow & & \downarrow \\
\begin{array}{c}(W_i^{-TW_i}, \Sigma^{-1}\partial W_i^{-T\partial W_i}) \\ \wedge \mathrm{map}(W, BG)_+\end{array} & \xrightarrow{\pi_i} & \begin{array}{c}(W_i^{-TW_i}, \Sigma^{-1}\partial W_i^{-T\partial W_i}) \\ \wedge \mathrm{map}(W_i, BG)_+\end{array} \\
& {}_{\mathrm{ev}}\searrow \quad \swarrow_{\mathrm{ev}} & \\
& \Sigma^\infty_+ BG & \\
& \downarrow \mathfrak{g} & \\
& ko &
\end{array}$$

the composite map from the top left hand corner to ko along the top and the right hand side of the diagram is $\pi_i^* \mathcal{T}^{\mathrm{cl}}_{\mathfrak{g},W_i}$, so the composite along the left hand side of the diagram also provides a classifying map for $\pi_i^* \mathcal{T}_{\mathfrak{g},W_i}$. Now the claim follows from the observation that the map

$$S \to (W^{-TW}, \Sigma^{-1}\partial W^{-T\partial W})$$

is equivalent to the composite

$$S \to \bigvee^{k} S \to \bigvee_{i=1}^{k} (W_i^{-TW_i}, \Sigma^{-1}\partial W_i^{-T\partial W_i})$$

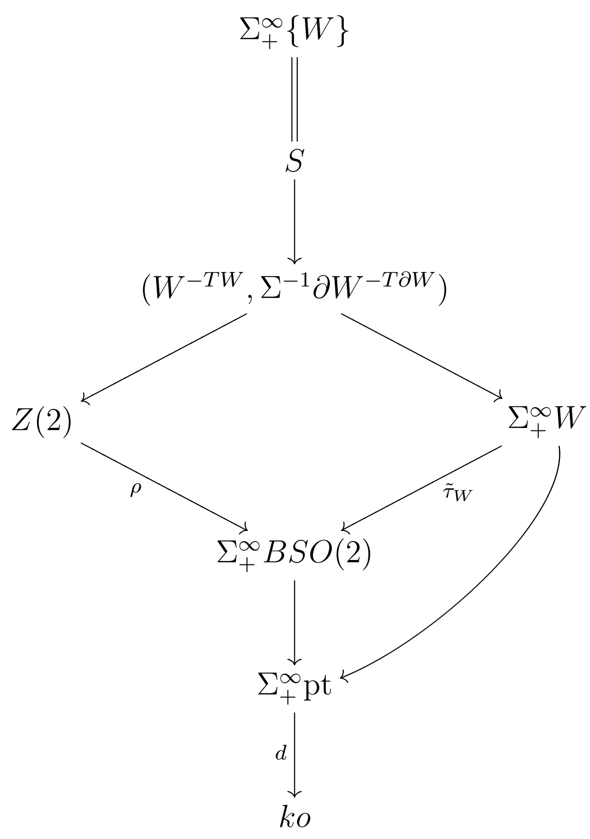
where the first map is the k-fold pinch map and the second map is the wedge of the maps

$$S \to (W_i^{-TW_i}, \Sigma^{-1}\partial W_i^{-T\partial W_i}),$$

$i = 1, \ldots, k$. □

Lemma 40. *The spectrum* $\mathcal{T}'_{d,W}$ *over the point* $W \in \operatorname{Mor} \mathcal{C}^+_{2,\partial}$ *is equivalent to* $S^{\chi(W)d}$*, so that the spectrum* $\mathcal{T}_{d,W}$ *over* $\operatorname{map}(W, BG)$ *is equivalent to* $S^{\chi(W)d}_{\operatorname{map}(W,BG)}$.

Proof. We have the homotopy-commutative diagram



(5.52)

The composite from top to bottom along the left-hand route classifies $\mathcal{T}'_{d,W}$. On the other hand, from Spanier–Whitehead duality theory of manifolds, we know that the composite map

$$S \to (W^{-TW}, \Sigma^{-1}\partial W^{-T\partial W}) \to \Sigma^\infty_+ W \to \Sigma^\infty_+\mathrm{pt} \simeq S$$

contained in path from top to bottom along the right-hand route is the map of degree $\chi(W)$. The claim follows. □

We are now ready to construct the field-theory operation associated with a morphism W of $\mathcal{C}^+_{2,\partial}$. The following theorem is the main result of the second half of this thesis.

Theorem 41. *Suppose $M_i = (M_i, a_i, \tilde{\tau}_{M_i}, \mathcal{O}_{M_i})$, $i = 0, 1$, are objects of $\mathcal{C}^+_{2,\partial}$. Then a non-identity morphism $W = (W, a_0, a_1, \tilde{\tau}_W, \mathcal{O}_W)$ from M_0 to M_1 induces a map*

$$H_\bullet(\mathrm{map}(M_0, BG);\, \mathcal{E}_{M_0}) \to H_{\bullet - d\chi(W)}(\mathrm{map}(M_1, BG);\, \mathcal{E}_{M_1}) \tag{5.53}$$

in the homotopy category of R-modules.

Proof. The source and target maps in the category $\mathcal{C}^+_{2,\partial}(BG)$ restrict to give us the diagram

$$\mathrm{map}(M_0, BG) \xleftarrow{s} \mathrm{map}(W, BG) \xrightarrow{t} \mathrm{map}(M_1, BG). \tag{5.54}$$

The map (5.53) is now the composite

$$\begin{aligned} H_\bullet(\mathrm{map}(M_0, BG);\, \mathcal{E}_{M_0}) &\xrightarrow{s^{\leftarrow}} H_\bullet(\mathrm{map}(W, BG);\, s^*\mathcal{E}_{M_0} \wedge_{\mathrm{map}(W,BG)} \mathcal{T}_{\mathfrak{g},W}) \\ &\simeq H_\bullet(\mathrm{map}(W, BG);\, t^*\mathcal{E}_{M_1} \wedge_{\mathrm{map}(W,BG)} \mathcal{T}_{d,W}) \\ &\simeq H_{\bullet - \chi(W)d}(\mathrm{map}(W, BG);\, t^*\mathcal{E}_{M_1}) \\ &\xrightarrow{t_\sharp} H_{\bullet - \chi(W)d}(\mathrm{map}(M_1, BG);\, \mathcal{E}_{M_1}). \end{aligned}$$

Here the map $s^{\leftarrow}$ exists by Lemma 39. The first equivalence follows from equation (5.50), and the second one from Lemma 40. □

5.5 Connections and conjectures

Having constructed the field-theory operation associated with a single cobordism in Theorem 41, in this section we will discuss conjectures that would take us closer to the construction of the Homological Conformal Field Theories we were referring to in Conjecture 1. In particular, we will show how the homological conformal field-theory operations would easily arise from the existence of a map of spectra over $\operatorname{Mor}\mathcal{C}^+_{2,\partial}$ which restricts to the operations of Theorem 41 on fibers. In addition, we will briefly discuss the connection of our conjectural field theories to Chataur and Menichi's [CM07] and Freed, Hopkins and Teleman's [FHT07a] work.

While our construction of the map (5.53) is not natural enough to ensure that the maps (5.53) for varying W fit together to give a map of spectra over $\operatorname{Mor}\mathcal{C}^+_{2,\partial}$, it is natural to conjecture that with a sufficiently careful construction, this would be the case. The following diagram displays the relevant spaces and maps. The triangles at the bottom left and right hand corners of the diagram are pullback squares, and the diagram (5.54) featuring in the construction of the field-theory operations of Theorem 41 is the fiber over $W \in \operatorname{Mor}\mathcal{C}^+_{2,\partial}$ of the upper part of the central diamond.

$$\begin{array}{ccccccc}
\operatorname{Obj}\mathcal{C}^+_{2,\partial}(BG) & \xleftarrow{\;s\;} & & \operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG) & & \xrightarrow{\;t\;} & \operatorname{Obj}\mathcal{C}^+_{2,\partial}(BG) \\
 & \nwarrow^{\mathrm{pr}_0} & \swarrow^{\bar{s}} & & \searrow^{\bar{t}} & \nearrow^{\mathrm{pr}_1} & \\
\downarrow U & & s^*\operatorname{Obj}\mathcal{C}^+_{2,\partial}(BG) & \downarrow U & t^*\operatorname{Obj}\mathcal{C}^+_{2,\partial}(BG) & & \downarrow U \\
 & & \searrow^{\pi_0} & & \swarrow^{\pi_1} & & \\
\operatorname{Obj}\mathcal{C}^+_{2,\partial} & \xleftarrow{\;s\;} & & \operatorname{Mor}\mathcal{C}^+_{2,\partial} & & \xrightarrow{\;t\;} & \operatorname{Obj}\mathcal{C}^+_{2,\partial}
\end{array} \tag{5.55}$$

The chief missing piece in the construction of the conjectural field theory operation over the space $\operatorname{Mor}\mathcal{C}^+_{2,\partial}$ is the identification of the spectrum $\mathcal{T}_{\mathfrak{g}}$ over $\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG)$ as the twisting associated with the pretransfer map induced by the map $\bar{s}$ in diagram (5.55). Moreover, while not strictly necessary for the construction of the operation, it would be important to identify the homotopy type of the spectrum $\mathcal{T}_d$ over $\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG)$ as well. The following conjecture is a strengthening of Lemma 39.

Conjecture 42. *The spectrum* $\mathcal{T}_{\mathfrak{g}}$ *over the space* $\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG)$ *is a Costenoble–Waner* $s^*\operatorname{Obj}\mathcal{C}^+_{2,\partial}(BG)$*-dual of* $S_{\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG)}$.

As for the spectrum $\mathcal{T}_d$ over $\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG)$, according to Lemma 40, the restrictions of $\mathcal{T}_d$ to the fibers of the map

$$\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG) \xrightarrow{U} \operatorname{Mor}\mathcal{C}^+_{2,\partial},$$

are trivial, but we do not expect $\mathcal{T}_d$ to be trivial over the whole space $\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG)$. Let $\mathscr{H}$ be the virtual vector bundle over $\operatorname{Mor}\mathcal{C}^+_{2,\partial}$ whose fiber over a non-identity morphism W is

$$\chi H_*(W,\partial_0 W;\, \mathbf{R}^d) = -H_1(W,\partial_0 W;\, \mathbf{R}^d)$$

where χ indicates the difference of the even- and odd-dimensional parts of the graded vector space $H_*(W,\partial_0 W;\, \mathbf{R}^d)$; the displayed equality then follows by the assumption that every connected component of W has both incoming and outgoing boundary components. Over the identity morphisms, we take $\mathscr{H}$ to be the zero bundle. The discussions in [CM07], Section 11, and [FHT07a], Section 4, suggest the following generalization of Lemma 40.

Conjecture 43. *The spectrum $\mathcal{T}'_d$ over $\operatorname{Mor}\mathcal{C}^+_{2,\partial}$ is equivalent to $S^{\mathscr{H}}_{\operatorname{Mor}\mathcal{C}^+_{2,\partial}}$, so that $\mathcal{T}_d \simeq U^* S^{\mathscr{H}}_{\operatorname{Mor}\mathcal{C}^+_{2,\partial}}$.*

Assuming that Conjecture 42 holds, we can construct the field-theory operation over $\operatorname{Mor}\mathcal{C}^+_{2,\partial}$ by a pull-push construction in the upper part of the central diamond in diagram (5.55). More precisely, we get the following map (in the homotopy category of R-modules over $\operatorname{Mor}\mathcal{C}^+_{2,\partial}$).

$$\begin{aligned}
\mathcal{H}_\bullet(s^*\operatorname{Obj}\mathcal{C}^+_{2,\partial}(BG);\, \operatorname{pr}_0^*\mathcal{E}) &\xrightarrow{\bar{s}^{\leftarrow}} \mathcal{H}_\bullet(\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG);\, \bar{s}^*\operatorname{pr}_0^*\mathcal{E} \wedge_{\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG)} \mathcal{T}_{\mathfrak{g}}) \\
&\simeq \mathcal{H}_\bullet(\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG);\, s^*\mathcal{E} \wedge_{\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG)} \mathcal{T}_{\mathfrak{g}}) \\
&\simeq \mathcal{H}_\bullet(\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG);\, t^*\mathcal{E} \wedge_{\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG)} \mathcal{T}_d) \\
&\simeq \mathcal{H}_\bullet(\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG);\, t^*\mathcal{E} \wedge_{\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG)} U^*\mathcal{T}'_d) \qquad (5.56)\\
&\simeq \mathcal{H}_\bullet(\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG);\, t^*\mathcal{E}) \wedge_{\operatorname{Mor}\mathcal{C}^+_{2,\partial}} \mathcal{T}'_d \\
&\simeq \mathcal{H}_\bullet(\operatorname{Mor}\mathcal{C}^+_{2,\partial}(BG);\, \bar{t}^*\operatorname{pr}_1^*\mathcal{E}) \wedge_{\operatorname{Mor}\mathcal{C}^+_{2,\partial}} \mathcal{T}'_d \\
&\xrightarrow{\bar{t}_\sharp} \mathcal{H}_\bullet(t^*\operatorname{Obj}\mathcal{C}^+_{2,\partial}(BG);\, \operatorname{pr}_1^*\mathcal{E}) \wedge_{\operatorname{Mor}\mathcal{C}^+_{2,\partial}} \mathcal{T}'_d
\end{aligned}$$

Here the umkehr map $\bar{s}^{\leftarrow}$ exists by Conjecture 42. The second equivalence from the top follows from equation (5.48), and the second equivalence from the bottom is a consequence of the projection formula.

We will now explain how the map (5.56) would give rise to the kind of operations one would expect from a Homological Conformal Field Theory. Let $M_i = (M_i, a_i, \tilde{\tau}_{M_i}, \mathcal{O}_{M_i})$, $i = 0, 1$, be objects of $\mathcal{C}^+_{2,\partial}$. Then under the equivalence of the analogue of the equation (5.2) for $\mathcal{C}^+_{2,\partial}$, the restriction of the diagram (5.55) from $\operatorname{Mor}\mathcal{C}^+_{2,\partial}$ to $\mathcal{C}^+_{2,\partial}(M_0, M_1)$ gives the diagram

$$\begin{array}{ccccc}
\operatorname{map}(M_0, BG) & \xleftarrow{s} & \coprod_{[W]} E\mathrm{Diff}^+(W;\partial W) \times_{\mathrm{Diff}^+(W;\partial W)} \operatorname{map}(W, BG) & \xrightarrow{t} & \operatorname{map}(M_1, BG) \\
\uparrow{\scriptstyle \mathrm{pr}_0} & \swarrow{\scriptstyle \bar{s}} & & \searrow{\scriptstyle \bar{t}} & \uparrow{\scriptstyle \mathrm{pr}_1} \\
& \coprod_{[W]} B\mathrm{Diff}^+(W;\partial W) \times \operatorname{map}(M_0, BG) & \downarrow & \coprod_{[W]} B\mathrm{Diff}^+(W;\partial W) \times \operatorname{map}(M_1, BG) & \\
\downarrow & \searrow{\scriptstyle \pi_0} & & \swarrow{\scriptstyle \pi_1} & \downarrow \\
\{M_0\} & \xleftarrow{s} & \coprod_{[W]} B\mathrm{Diff}^+(W;\partial W) & \xrightarrow{t} & \{M_1\}
\end{array} \tag{5.57}$$

and the restriction of the map (5.56) the map (in the homotopy category of R-modules over $\coprod_{[W]} B\mathrm{Diff}^+(W;\partial W)$)

$$\begin{aligned}
&\mathcal{H}_\bullet\Big(\coprod_{[W]} B\mathrm{Diff}^+(W;\partial W) \times \operatorname{map}(M_0, BG);\ \mathrm{pr}_0^*\mathcal{E}_{M_0}\Big) \\
\to &\mathcal{H}_\bullet\Big(\coprod_{[W]} B\mathrm{Diff}^+(W;\partial W) \times \operatorname{map}(M_1, BG);\ \mathrm{pr}_1^*\mathcal{E}_{M_1}\Big) \wedge_{\coprod_{[W]} B\mathrm{Diff}^+(W;\partial W)} \mathcal{T}''_d
\end{aligned} \tag{5.58}$$

where $\mathcal{T}''_d$ denotes the pullback of $\mathcal{T}'_d$ to $\coprod_{[W]} B\mathrm{Diff}^+(W;\partial W)$. Applying the functor

$$r_!\left(- \wedge_{\coprod_{[W]} B\mathrm{Diff}^+(W;\partial W)} (\mathcal{T}''_d)^{-1}\right),$$

where r denotes the constant map to the one-point space, and using the projection formula, we obtain a map

$$\begin{aligned}
&H_\bullet\Big(\coprod_{[W]} B\mathrm{Diff}^+(W;\partial W) \times \operatorname{map}(M_0, BG);\ \mathrm{pr}_0^*\mathcal{E}_{M_0} \wedge_{(..)} \pi_0^*(\mathcal{T}''_d)^{-1}\Big) \\
\to &H_\bullet\Big(\coprod_{[W]} B\mathrm{Diff}^+(W;\partial W) \times \operatorname{map}(M_0, BG);\ \mathrm{pr}_1^*\mathcal{E}_{M_1}\Big).
\end{aligned} \tag{5.59}$$

The domain of this map is equivalent to

$$H_\bullet\Big(\coprod_{[W]} B\mathrm{Diff}^+(W;\partial W);\,(\mathcal{T}_d'')^{-1}\Big)\wedge H_\bullet\Big(\mathrm{map}(M_0,BG);\,\mathcal{E}_{M_0}\Big)$$

while $(\mathrm{pr}_1)_\sharp$ gives a map from the codomain to the R-module

$$H_\bullet(\mathrm{map}(M_1,BG);\,\mathcal{E}_{M_1}).$$

Thus we get a map (in the homotopy category of R-modules)

$$\begin{aligned}&H_\bullet\Big(\coprod_{[W]} B\mathrm{Diff}^+(W;\partial W);\,(\mathcal{T}_d'')^{-1}\Big)\wedge H_\bullet\Big(\mathrm{map}(M_0,BG);\,\mathcal{E}_{M_0}\Big)\\ \to &H_\bullet(\mathrm{map}(M_1,BG);\,\mathcal{E}_{M_1}).\end{aligned}\tag{5.60}$$

It is these maps for varying M_0 and M_1 that give the Homological Conformal Field Theory operations we had in mind in Conjecture 1.

We will now comment briefly on the connection of the HCFT operation 5.60 to the Chataur–Menichi string topology of BG; this will also be an opportunity for us to discuss the twisting $(\mathcal{T}_d'')^{-1}$ appearing as the coefficients of $\coprod_{[W]} B\mathrm{Diff}^+(W;\partial W)$ in 5.60. In [CM07], given a field k and a cobordism W, Chataur and Menichi construct HCFT operations

$$\begin{aligned}&(\det H_1(W,\partial_0 W;\,\mathbf{Z}))^{\otimes d}\otimes_{\mathbf{Z}} H_*(B\mathrm{Diff}^+(W;\partial W);\,k)\otimes_k H_*(\mathrm{map}(\partial_0 W,BG);\,k)\\ \to &H_*(\mathrm{map}(\partial_1 0W,BG);\,k)\end{aligned}\tag{5.61}$$

where the factor $(\det H_1(W,\partial_0 W;\,\mathbf{Z}))^{\otimes d}$ is to be thought of as having degree $-d\chi(W)$. In terms of universal R-orientations, these operations would correspond to a case where R is the Eilenberg–Mac Lane spectrum Hk, and the map ε in diagram (5.46) is zero. Thus such a universal orientation would simply correspond to a null homotopy of the composite

$$\Sigma^\infty_+ BSO(2)\wedge BG\xrightarrow{\bar\sigma_{\mathfrak{g}}} ko\to \mathrm{line}_\bullet(R).$$

Recalling the definition of $\bar\sigma_{\mathfrak{g}}$, we see that a choice of an orientation of the vector bundle $\mathrm{ad}(EG)$ over BG is enough to induce such a null homotopy. Notice that for G connected the vector bundle $\mathrm{ad}(EG)$ is always orientable, for then the adjoint representation of G factors through $SO(d)$, but $\mathrm{ad}(EG)$ may fail to be orientable in general. Indeed, Chataur and Menichi assume that G is connected in their work.

Let us denote by χ_d the vector bundle

$$\coprod_{[W]} E\mathrm{Diff}^+(W;\partial W) \times_{\mathrm{Diff}^+(W;\partial W)} H_1(W, \partial_0 W;\, \mathbf{R}^d) \to \coprod_{[W]} B\mathrm{Diff}^+(W;\partial W),$$

and let us assume from now on that Conjecture 43 is correct. Then

$$(\mathcal{T}_d'')^{-1} = \Sigma^\infty_{\coprod_{[W]} B\mathrm{Diff}^+(W;\partial W)} S^{\chi_d}_{\coprod_{[W]}(..)}.$$

It follows from [God07], Lemma 20 that χ_d is orientable, so that

$$(\mathcal{T}_d'')^{-1} \wedge H\mathbf{Z}$$

is a trivializable $H\mathbf{Z}$-line bundle. A choice of a trivialization is equivalent to the choice of an orientation for χ_d, and over a single component $B\mathrm{Diff}^+(W;\partial W)$ of $\coprod_{[W]} B\mathrm{Diff}^+(W;\partial W)$, an orientation of χ_d corresponds to the choice of a generator for the $\mathbf{Z}$-module $(\det H_1(W, \partial W; \mathbf{Z}))^{\otimes d}$. Thus, with a universal Hk-orientation of the type we discussed above, upon passing to homotopy groups, the map (5.60) would give rise to a map as in (5.61), where the factor $(\det H_1(W, \partial W; \mathbf{Z}))^{\otimes d}$ encodes a choice of trivialization of the spectrum $(\mathcal{T}_d'')^{-1} \wedge H\mathbf{Z}$ over $B\mathrm{Diff}^+(W;\partial W)$, and hence also of the spectrum $(\mathcal{T}_d'')^{-1} \wedge Hk$ over $B\mathrm{Diff}^+(W;\partial W)$.

Having discussed the connection of our field-theory operations to Chataur and Menichi's string topology of BG, we will now briefly comment on their relationship to Freed, Hopkins and Teleman's field-theory operations [FHT07a]. The classifying spectrum pic_g for geometric K-theory twistings considered by Freed, Hopkins and Teleman admits a map to $\mathrm{line}_\bullet(K)$, and hence a universal orientation in the sense of Freed, Hopkins and Teleman gives rise to a universal K-orientation. The definitions of Freed, Hopkins and Teleman's operations and ours are then parallel: the Freed–Hopkins–Teleman operation associated with a cobordism W arises from a pull–push construction in the correspondence diagram

$$\mathcal{M}_{\partial_0 W} \xleftarrow{s} \mathcal{M}_W \xrightarrow{t} \mathcal{M}_{\partial_1 W} \tag{5.62}$$

of moduli stacks of flat G-connections, while ours arises from a pull–push construction in the diagram

$$\mathrm{map}(\partial_0 W, BG) \xleftarrow{s} \mathrm{map}(W, BG) \xrightarrow{t} \mathrm{map}(\partial_1 W, BG) \tag{5.63}$$

obtained from diagram (5.62) by passing to classifying spaces of stacks. There is one complication in that Freed, Hopkins and Teleman work with K-cohomology and use a pretransfer map induced by the map t in the construction of their operation, while we have worked homologically and use pretransfer map induced by the map s. However, Poincaré duality for the stacks of (5.62) should help to bridge this gap.

We conclude by acknowledging two major omissions in the discussion of our field theories. First, we have not said anything about the functoriality of our operations. In terms of the map (5.56), the functoriality of the operations should be expressed in terms of a compatibility relation between the pullbacks of the map (5.56) over $\mathrm{Mor}\,\mathcal{C}^+_{2,\partial}$ under the three morphisms

$$\mathrm{Mor}\,\mathcal{C}^+_{2,\partial\, s} \times_{\mathrm{Obj}\,\mathcal{C}^+_{2,\partial}} {}_t\mathrm{Mor}\,\mathcal{C}^+_{2,\partial} \to \mathrm{Mor}\,\mathcal{C}^+_{2,\partial}$$

given by composition and the projections to the two factors. We will not spell out the details here. Second, field theories should be *symmetric monoidal* functors, and we have not touched this aspect of our operations at all. A prerequisite for the discussion of the monoidality of our field-theory operations would be a good understanding of the sense in which the cobordism categories $\mathcal{C}^+_{2,\partial}$ are symmetric monoidal categories. It seems to us that the symmetric monoidal structure on the category $\mathcal{C}^+_{2,\partial}$ should be best expressed by an action of an E_∞ operad on the spaces of objects and morphisms. Indeed, thinking of $\mathbf{R}^L$ as an open L-cube, the little L-cubes operad acts naturally on $\mathrm{Obj}\,\mathcal{C}^+_{2,L}$ and $\mathrm{Mor}\,\mathcal{C}^+_{2,L}$, and these actions are compatible as L goes to infinity. The monoidality of our operations should then be expressed as a compatibility of the map (5.56) with the E_∞ action on $\mathrm{Mor}\,\mathcal{C}^+_{2,\partial}$. Again, we leave the details for future work.

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